Digital Signal Processing

Lab 08: Fourier Analysis for CT Signals and Systems

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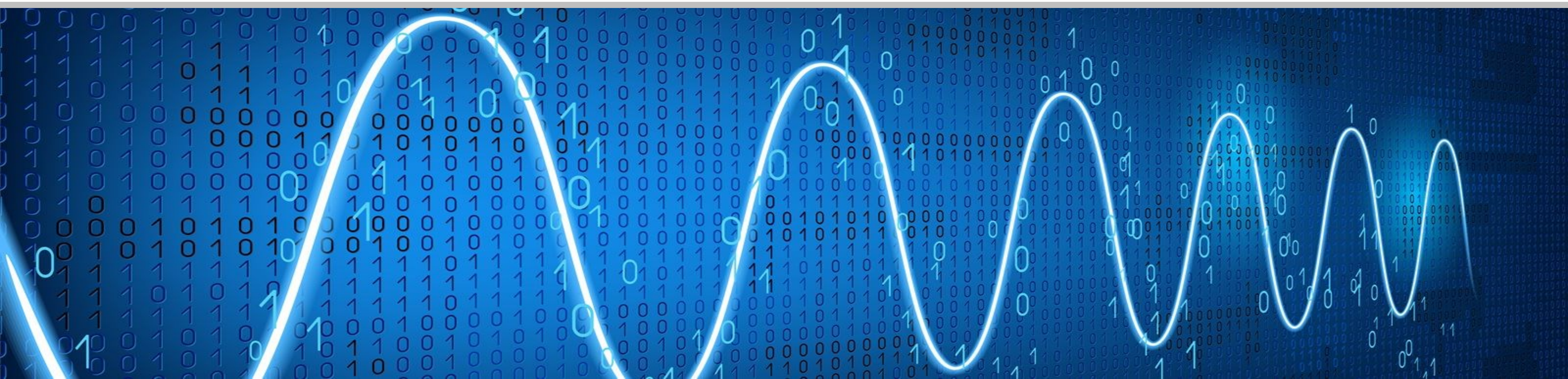


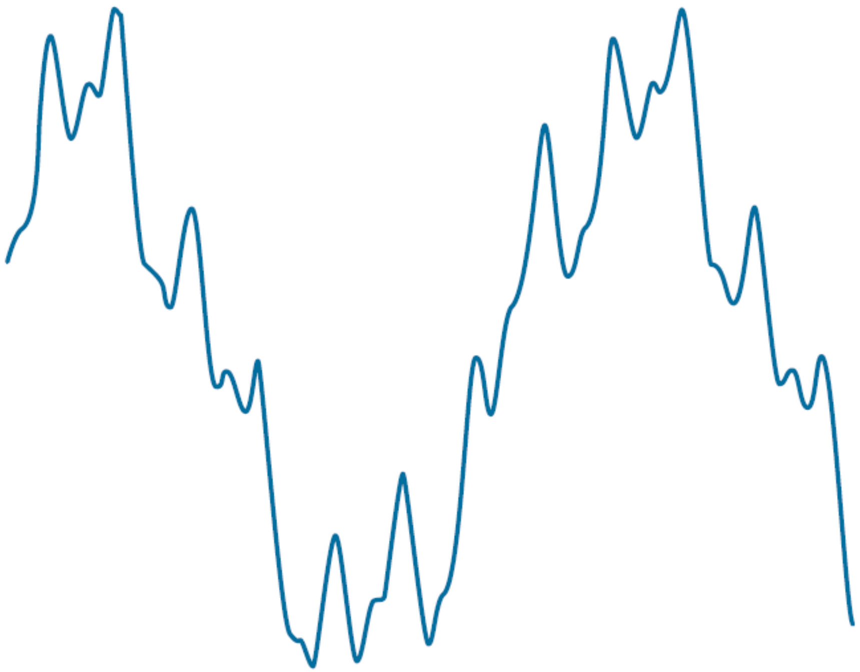
Fourier Analysis for Continuous-Time Signals and Systems

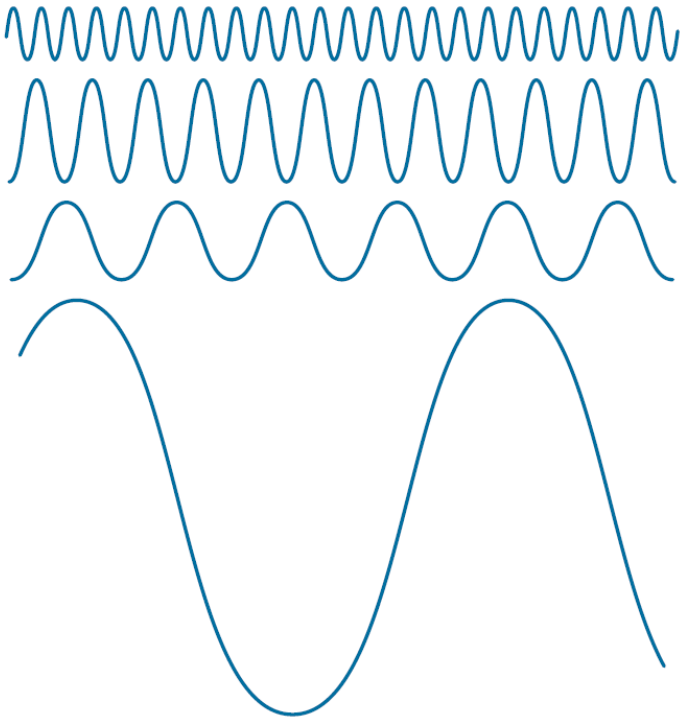
The purpose of this lab is to

- Learn the Fourier transform for non-periodic signals as an extension of Fourier series for periodic signals.
- Study properties of the Fourier transform.
- Understand energy and power spectral density concepts.

Fourier Series

- **Fourier Series**: Any periodic function can be expressed as the sum of sines and/or cosines of different frequencies, each multiplied by a different coefficient.



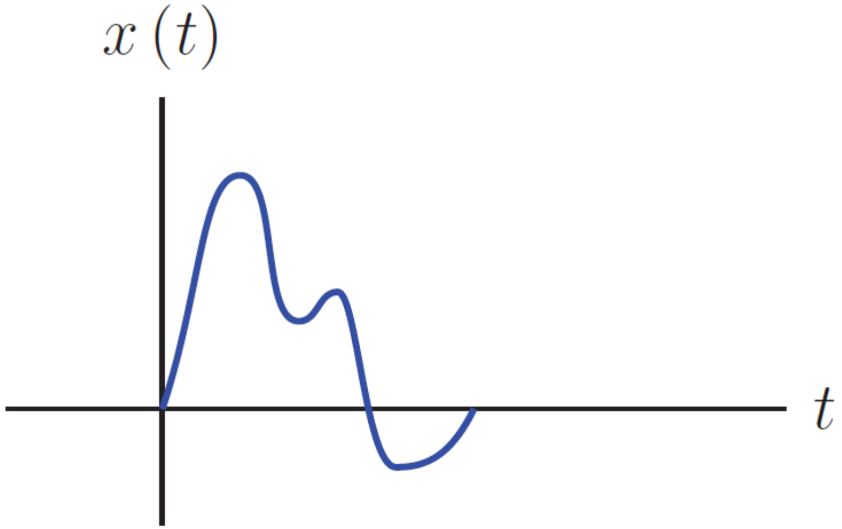


Analysis of Non-Periodic Continuous-Time Signals

- We must also realize that we often work with signals that are not necessarily periodic.
- We would like to have similar capability when we use non-periodic signals in conjunction with linear and time-invariant systems.
- These efforts will lead us to the Fourier transform for continuous-time signals.

Analysis of Non-Periodic Continuous-Time Signals

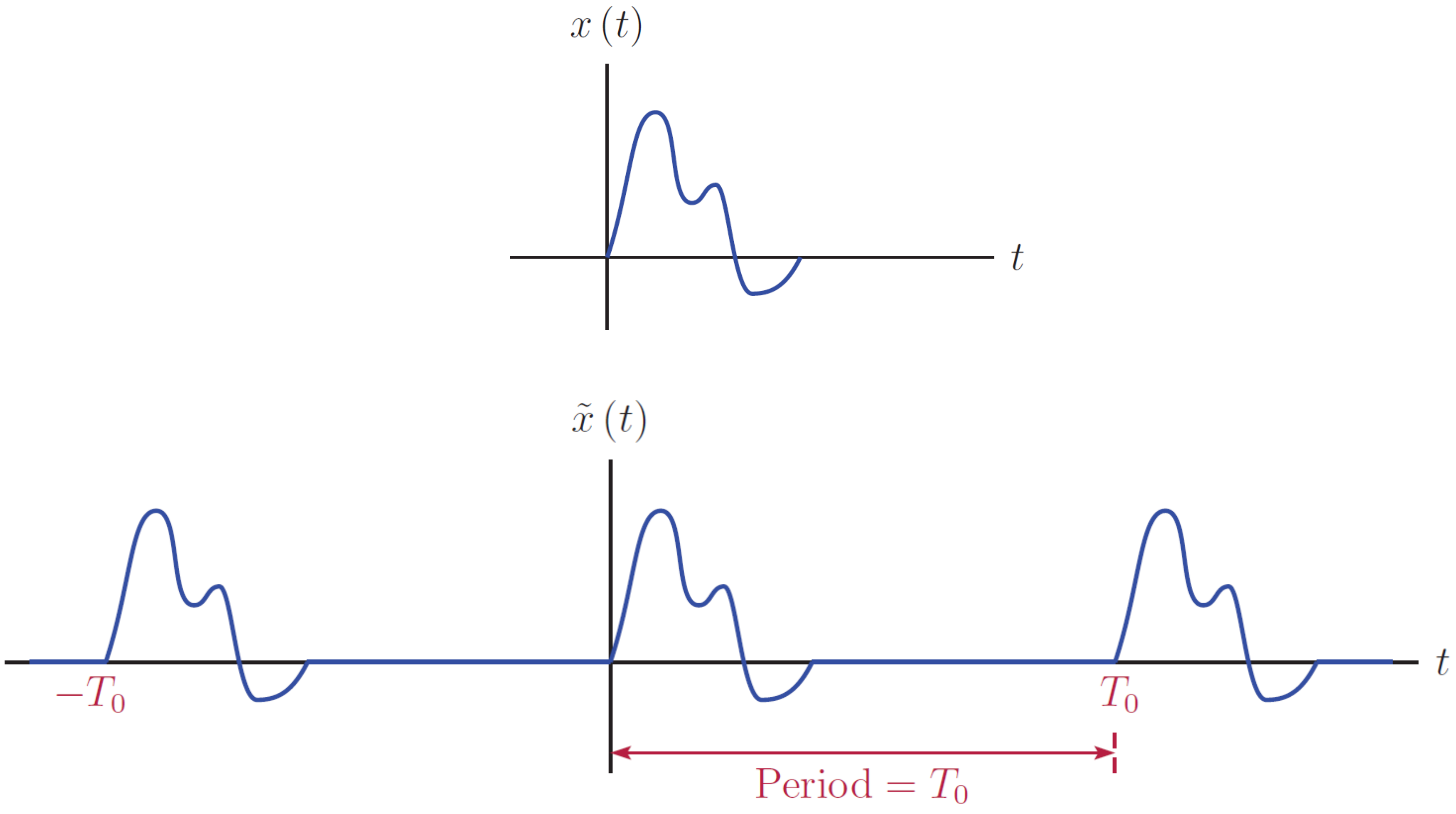
- Consider the non-periodic signal $x(t)$



- We already know how to represent periodic signals in the frequency domain.

Analysis of Non-Periodic Continuous-Time Signals

- Let us construct a periodic extension $\tilde{x}(t)$ of the signal $x(t)$ by repeating it at intervals of T_0.



Fourier Transform For Continuous-Time Signals

Fourier transform for continuous-time signals:

Analysis equation: (Forward transform)

$$X\left(\omega\right)=\int_{-\infty}^{\infty}x\left(t\right)\,e^{-j\omega t}\,dt$$

Synthesis equation: (Inverse transform)

$$x\left(t\right)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X\left(\omega\right)\,e^{j\omega t}\,d\omega$$

Fourier Transform For Continuous-Time Signals

Fourier transform for continuous-time signals (using f instead of ω):

Analysis equation: (Forward transform)

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

Synthesis equation: (Inverse transform)

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t}\, df$$

Analysis of Non-Periodic Continuous-Time Signals

- The forward transform

$$X(\omega) = \mathcal{F}\{x(t)\}$$

- The inverse transform

$$x(t) = \mathcal{F}^{-1}\{X(\omega)\}$$

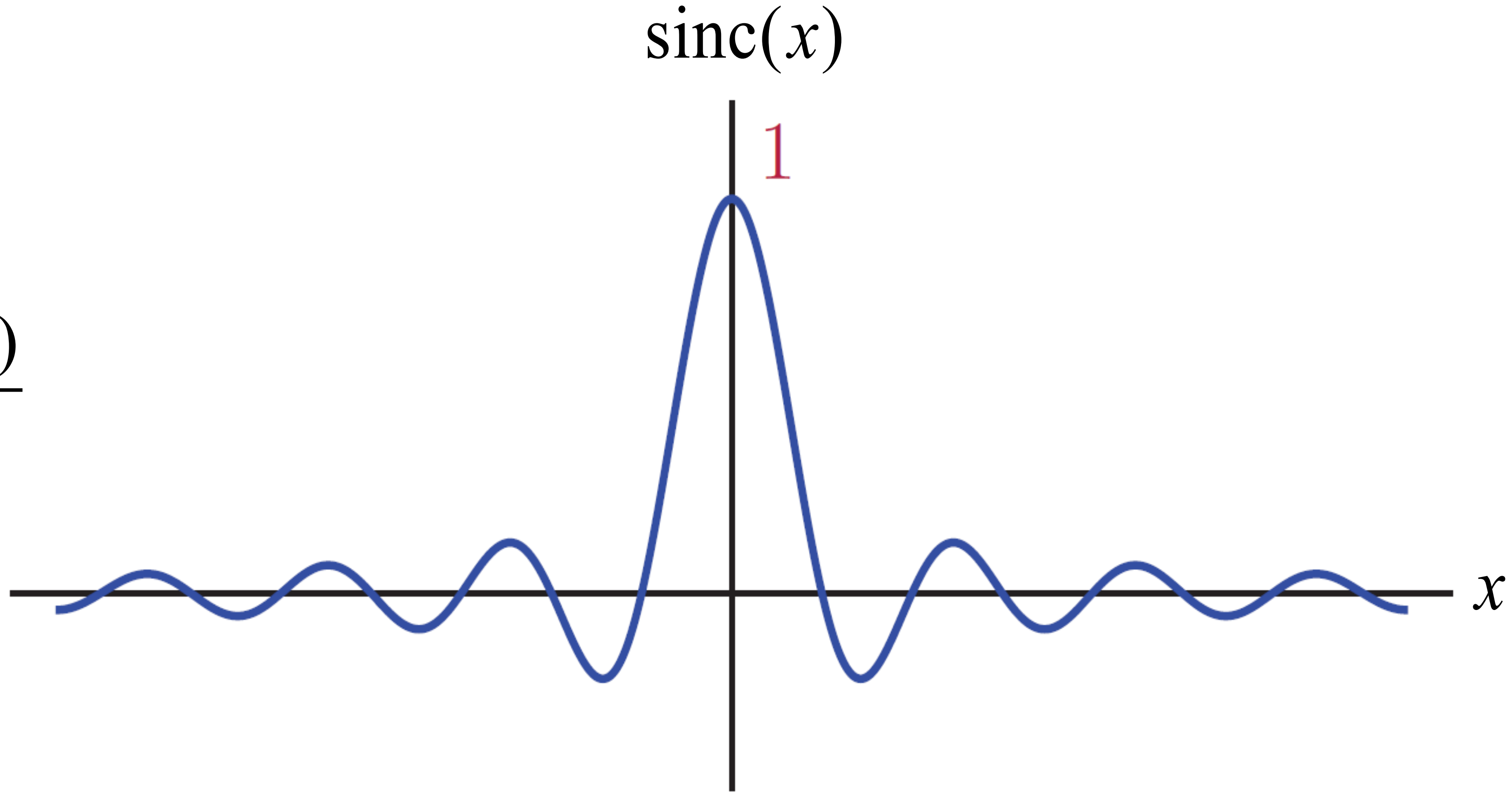
- The relationship between $x(t)$ and $X(\omega)$ is in the form

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$$

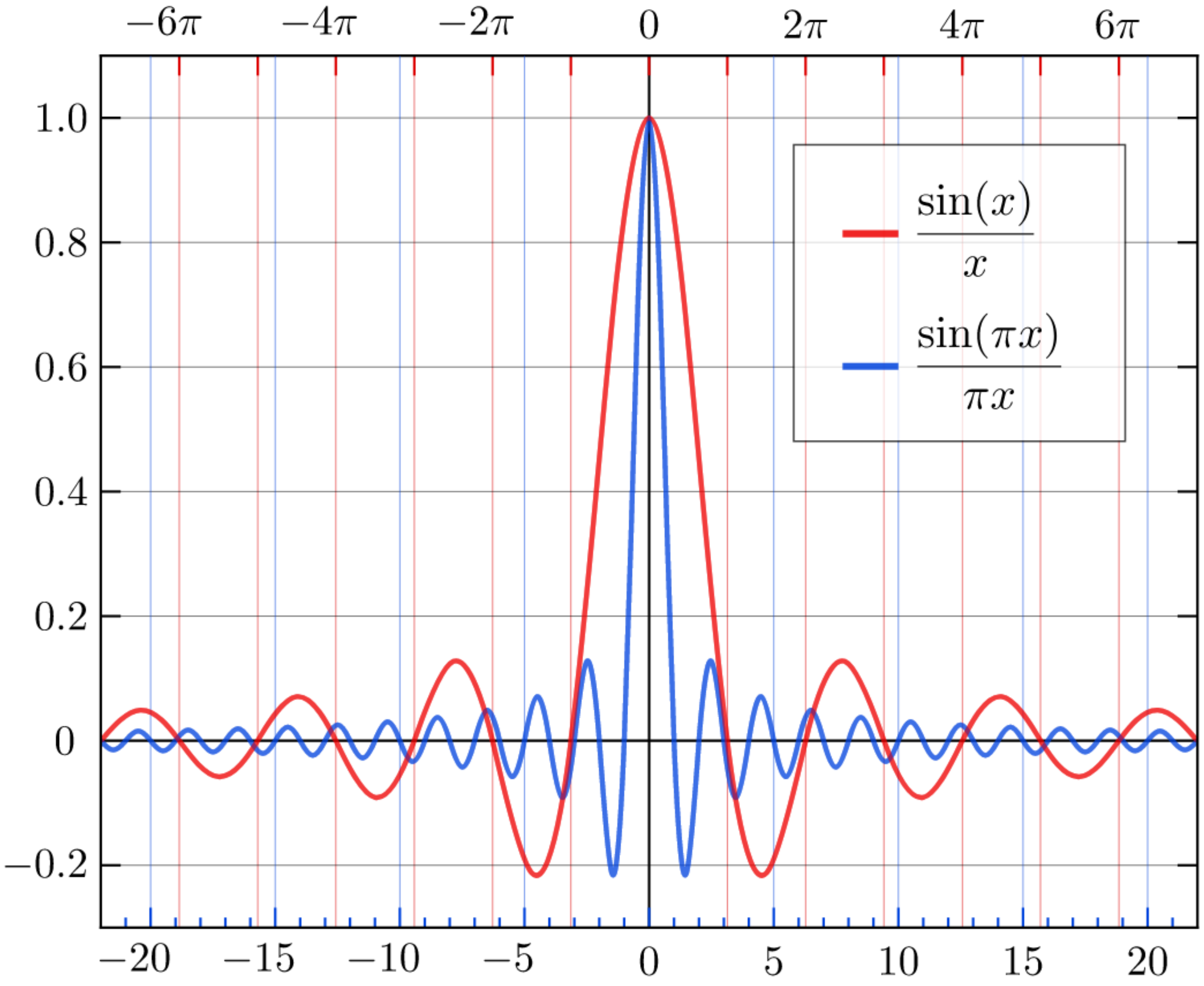
Sinc Function and Normalized Sinc Function

$$\mathrm{sinc}(x) = \frac{\sin(x)}{x}$$

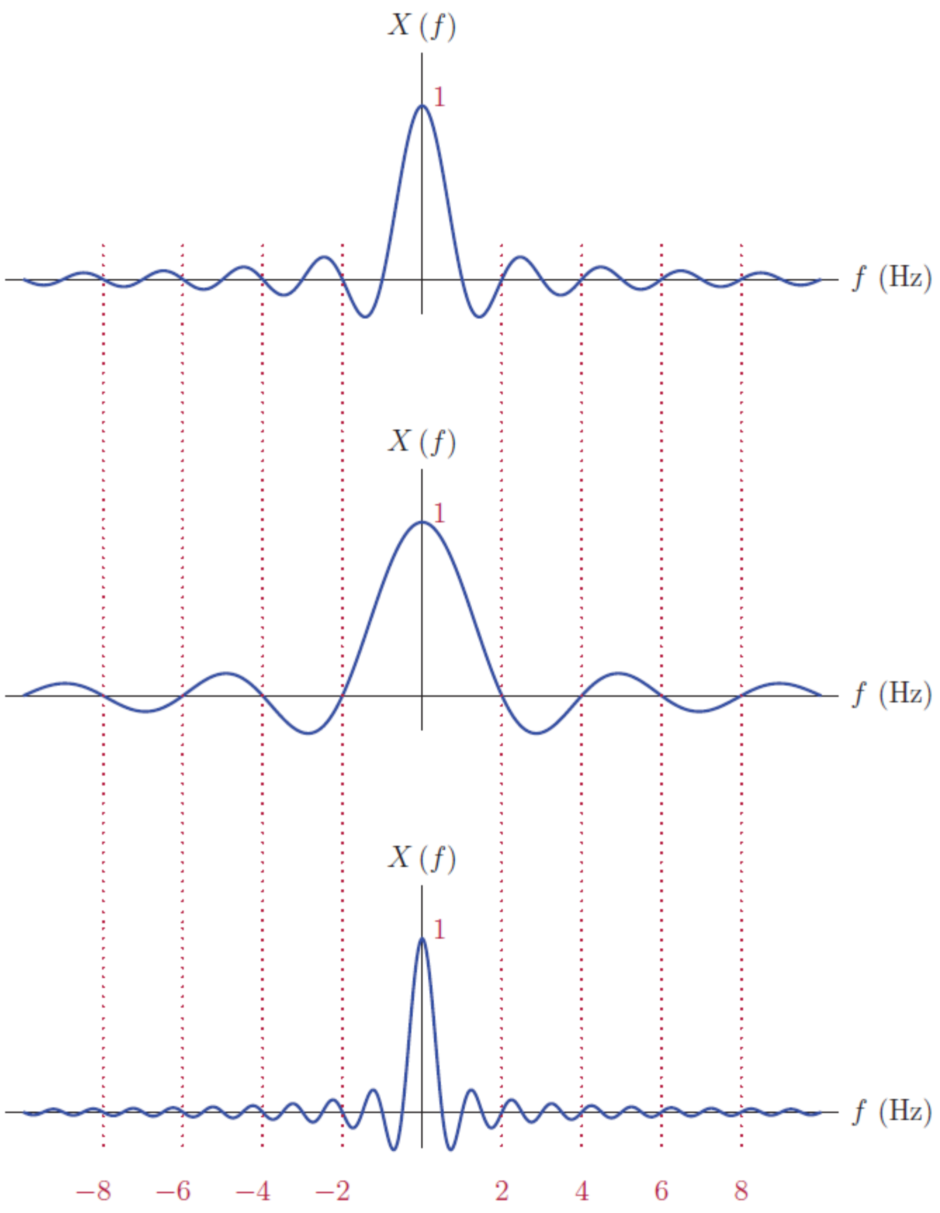
$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$



Sinc Function and Normalized Sinc Function



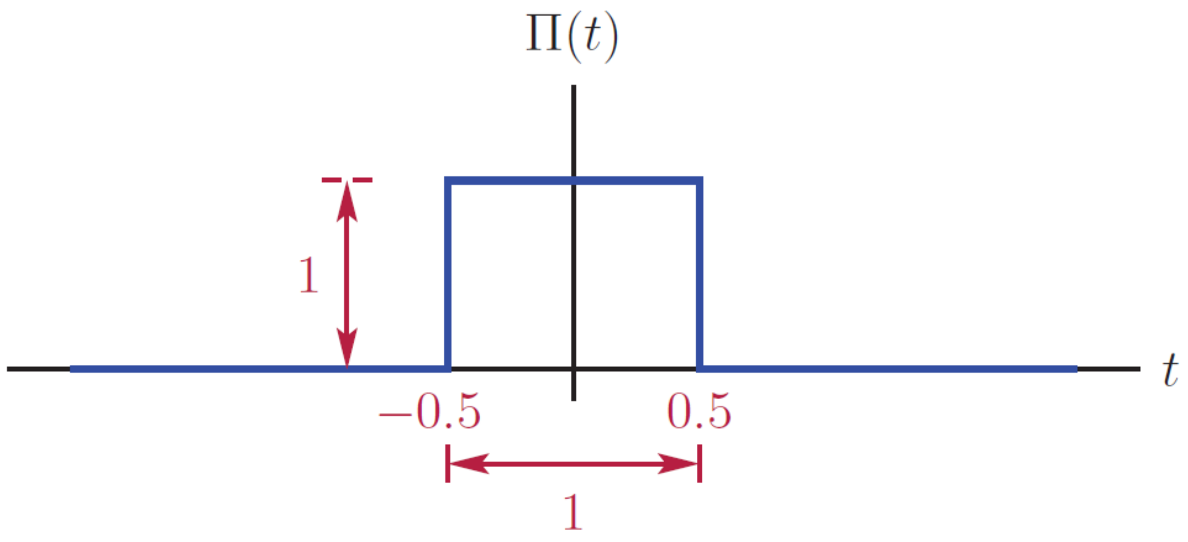
Sinc Function



Unit-Pulse Function

- We will define the unit-pulse function as a rectangular pulse with unit width and unit amplitude, centered around the origin.

$$\Pi(t) = \begin{cases} 1, & |t| < \frac{1}{2} \\ 0, & |t| > \frac{1}{2} \end{cases}$$



Example 4.12: Fourier Transform of a Rectangular Pulse

Example 4.12: Fourier transform of a rectangular pulse

Using the forward Fourier transform integral in Eqn. (4.127), find the Fourier transform of the isolated rectangular pulse signal

$$x(t) = A\,\Pi\left(\frac{t}{\tau}\right)$$

shown in Fig. 4.35.

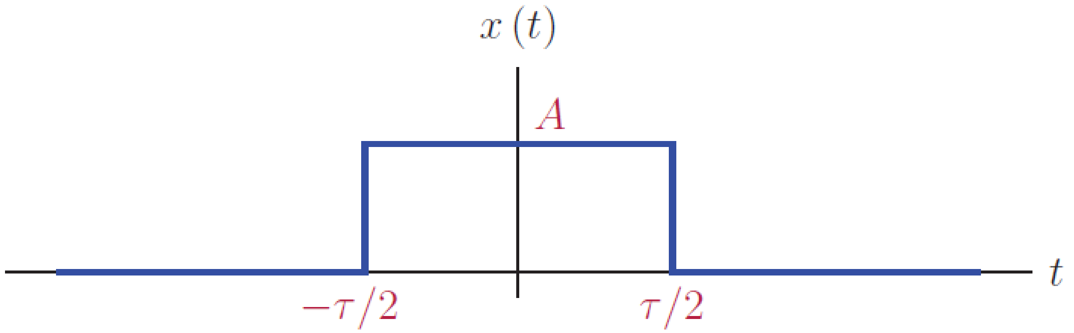


Figure 4.35 – Isolated pulse with amplitude A and width τ for Example 4.12.

Example 4.12 – Solution

$$X(\omega) = \int_{-\tau/2}^{\tau/2} (A) e^{-j\omega t} dt = A \frac{1}{-j\omega} e^{-j\omega t} \Big|_{-\tau/2}^{\tau/2}$$

$$= \frac{A}{-j\omega} \left[\cos(-\omega t) + j \sin(-\omega t) \right] \Big|_{-\tau/2}^{\tau/2} = \frac{A}{-j\omega} \left[\cos(\omega t) - j \sin(\omega t) \right] \Big|_{-\tau/2}^{\tau/2}$$

$$= \frac{A}{-j\omega} \left[0 - j \sin(\omega t) \right] \Big|_{-\tau/2}^{\tau/2} = \frac{A}{\omega} \sin(\omega t) \Big|_{-\tau/2}^{\tau/2}$$

$$= \frac{2A}{\omega} \sin\left(\frac{\omega\tau}{2} \right)$$

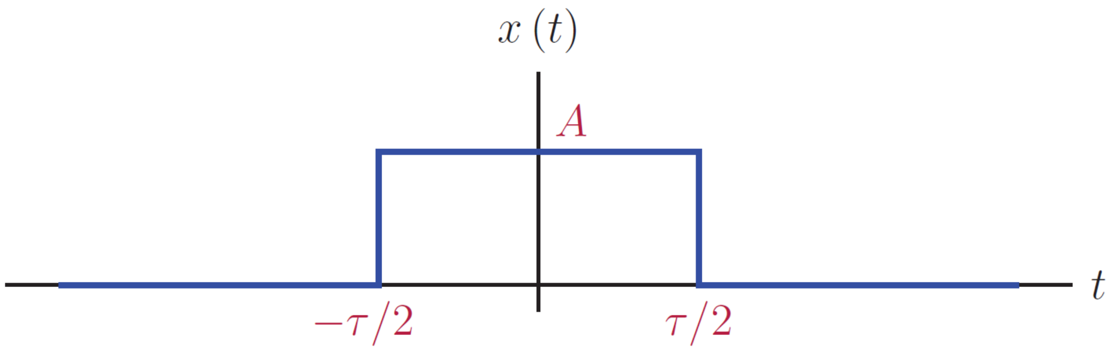
$$= \frac{2A\tau}{\omega\tau} \sin\left(\frac{\omega\tau}{2} \right) = A\tau \frac{2}{\omega\tau} \sin\left(\frac{\omega\tau}{2} \right)$$

$$= A\tau \ \mathrm{sinc}\left(\frac{\omega\tau}{2\pi} \right)$$

$$X(f) = A\tau \ \mathrm{sinc}\left(f\tau \right)$$

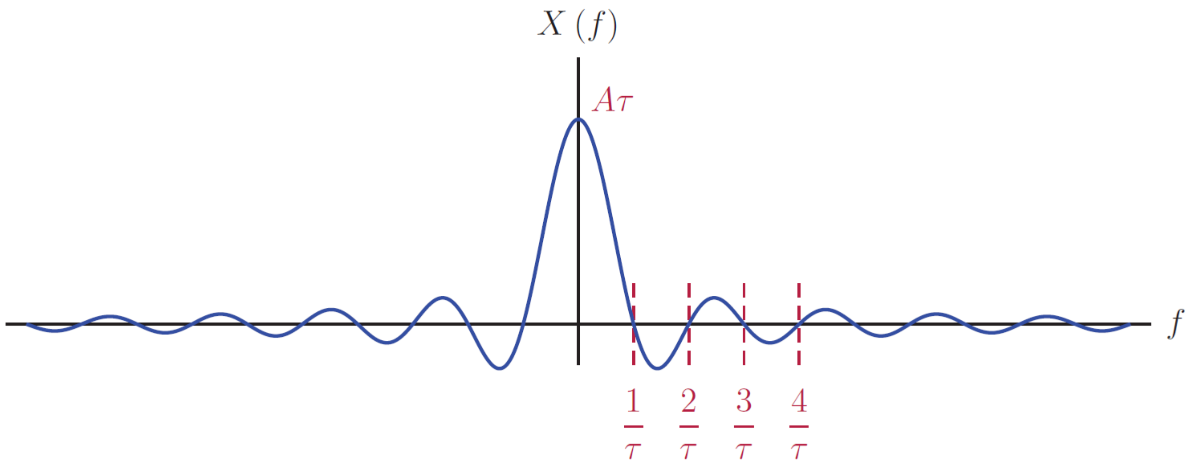
Example 4.12 – Solution

$$x(t) = A\prod\left(\frac{t}{\tau}\right)$$

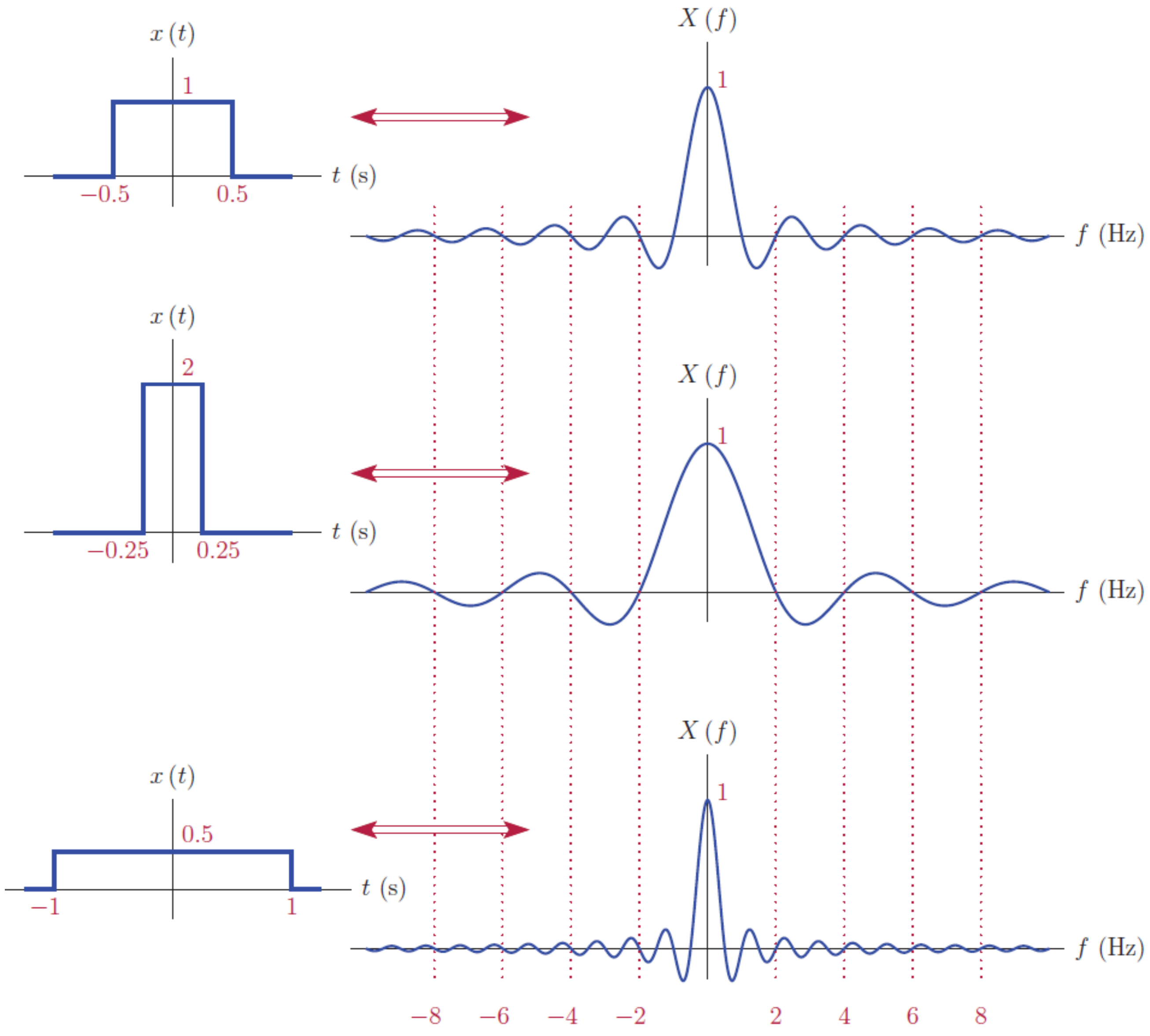


$$\mathcal{F}\{x(t)\} = X(\omega) = A\tau \,\operatorname{sinc}\left(\frac{\omega\tau}{2\pi}\right)$$

$$\mathcal{F}\{x(t)\} = X(f) = A\tau \,\operatorname{sinc}(f\tau)$$



Example 4.12 – Solution



Problem 4.18

4.18. Find the Fourier transform of each of the pulse signals given below:

a. $$x(t) = 3\,\Pi(t)$$

c. $$x(t) = 2\,\Pi\left(\frac{t}{4}\right)$$

Problem 4.18 (a) – Solution

a. $x(t) = 3\,\Pi(t)$

$$x(t) = A\,\Pi\left(\frac{t}{\tau}\right) \xrightarrow{\mathcal{F}} X(f) = A\tau\ \mathrm{sinc}(f\tau)$$

$$x(t) = 3\,\Pi(t) \xrightarrow{\mathcal{F}} X(f) = 3\ \mathrm{sinc}(f)$$

Problem 4.18 (c) – Solution

c. $$x(t) = 2\,\Pi\left(\frac{t}{4}\right)$$

$$x(t) = A\,\Pi\left(\frac{t}{\tau}\right) \xrightarrow{\mathcal{F}} X(f) = A\tau\ \text{sinc}(f\tau)$$

$$x(t) = 2\,\Pi\left(\frac{t}{4}\right) \xrightarrow{\mathcal{F}} X(f) = 8\ \text{sinc}(4f)$$

Example 4.14: Fourier Transform of the Unit-Impulse Function

Example 4.14: Transform of the unit-impulse function

The unit-impulse function was defined in Section 1.3.2 of Chapter 1. The Fourier transform of the unit-impulse signal can be found by direct application of the Fourier transform integral along with the sifting property of the unit-impulse function.

$$\mathcal{F}\left\{\delta(t)\right\} = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt = e^{-j\omega t}\Big|_{t=0} = 1$$

Example 4.15: Fourier Transform of a Right-Sided Exponential Signal

Example 4.15: **Fourier transform of a right-sided exponential signal**

Determine the Fourier transform of the right-sided exponential signal

$$x(t) = e^{-at}\, u(t)$$

with $a > 0$ as shown in Fig. 4.43.

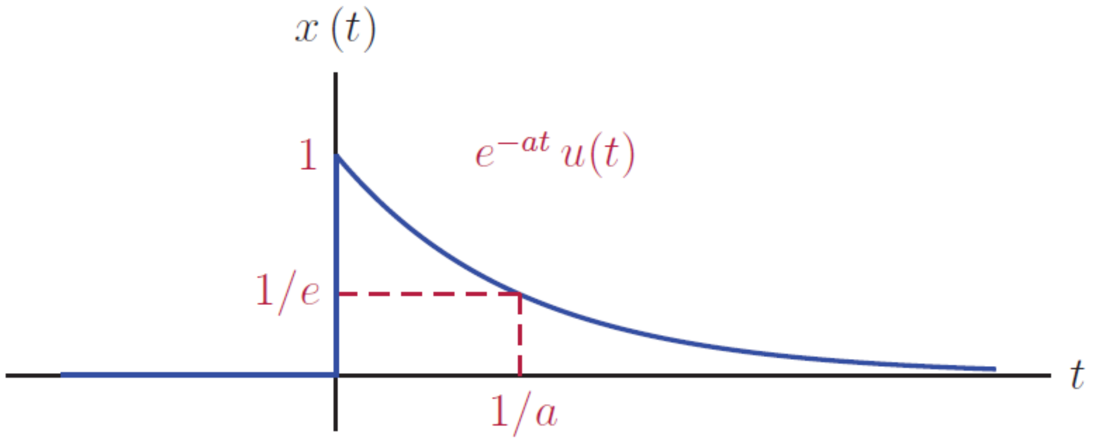


Figure 4.43 – Right-sided exponential signal for Example 4.15.

Example 4.15 – Solution

$$X(\omega) = \int_{-\infty}^{\infty} e^{-at} u(t) e^{-j\omega t} dt$$

$$= \int_{0}^{\infty} e^{-at} e^{-j\omega t} dt = \int_{0}^{\infty} e^{-at-j\omega t} dt$$

$$= \int_{0}^{\infty} e^{-t(a+j\omega)} dt = \frac{-1}{a+j\omega} e^{-t(a+j\omega)} \Big|_{0}^{\infty}$$

$$= \frac{-1}{a+j\omega} [0-1]$$

$$= \frac{1}{a+j\omega}$$

Example 4.15 – Book Solution

Solution: Application of the Fourier transform integral of Eqn. (4.127) to $x(t)$ yields

$$X(\omega) = \int_{-\infty}^{\infty} e^{-at}\, u(t)\, e^{-j\omega t}\, dt$$

Changing the lower limit of integral to $t = 0$ and dropping the factor $u(t)$ results in

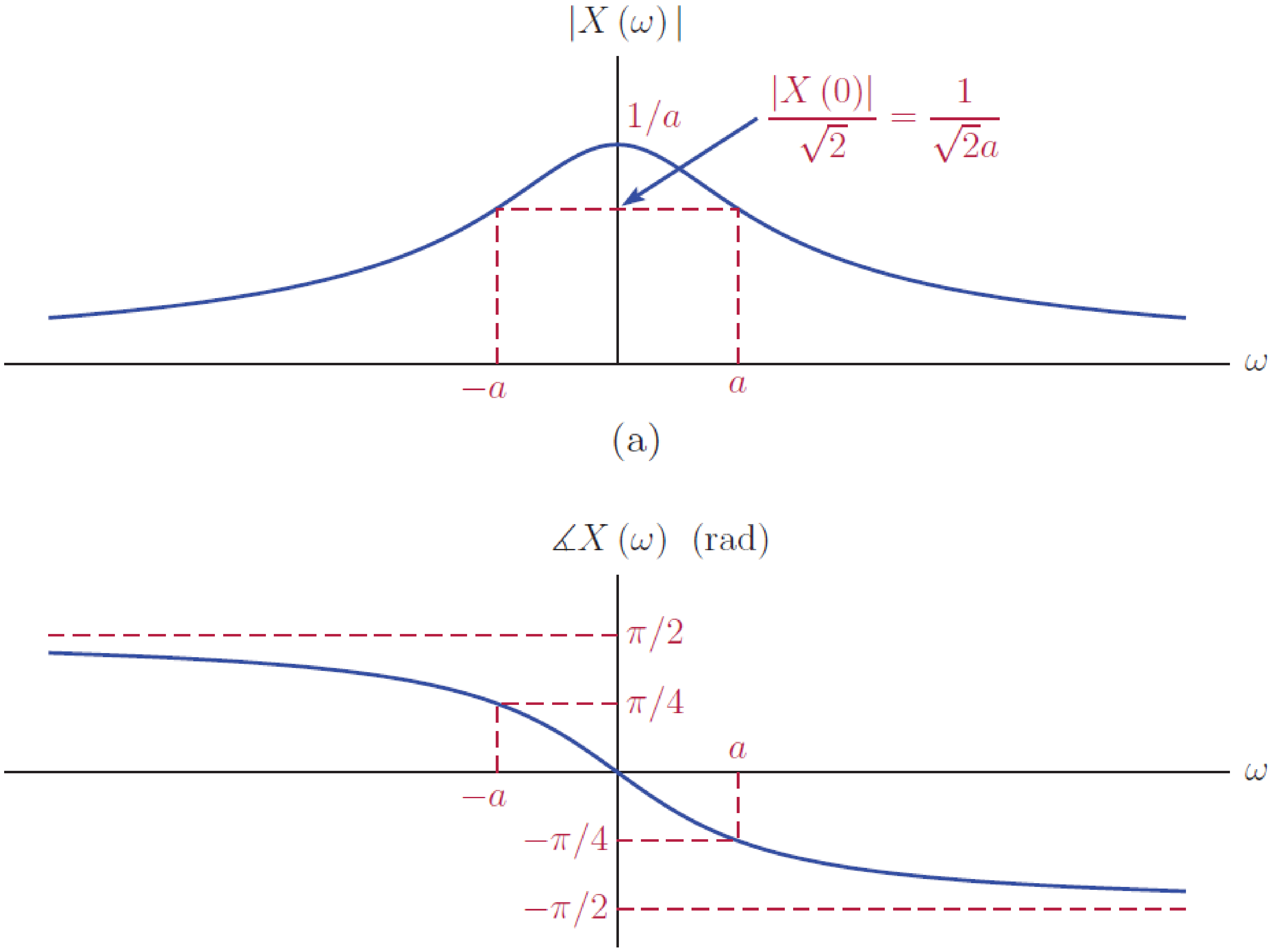
$$X(\omega) = \int_{0}^{\infty} e^{-at}\, e^{-j\omega t}\, dt = \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt = \frac{1}{a + j\omega}$$

This result in Eqn. (4.155) is only valid for $a > 0$ since the integral could not have been evaluated otherwise. The magnitude and the phase of the transform are

$$|X(\omega)| = \left|\frac{1}{a + j\omega}\right| = \frac{1}{\sqrt{a^2 + \omega^2}}$$

$$\theta(\omega) = \measuredangle X(\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$

Example 4.15 – Book Solution



Example 4.16: Fourier Transform of a Two-Sided Exponential Signal

Example 4.16: Fourier transform of a two-sided exponential signal

Determine the Fourier transform of the two-sided exponential signal given by

$$x(t) = e^{-a|t|}$$

where a is any non-negative real-valued constant. The signal $x(t)$ is shown in Fig. 4.46.

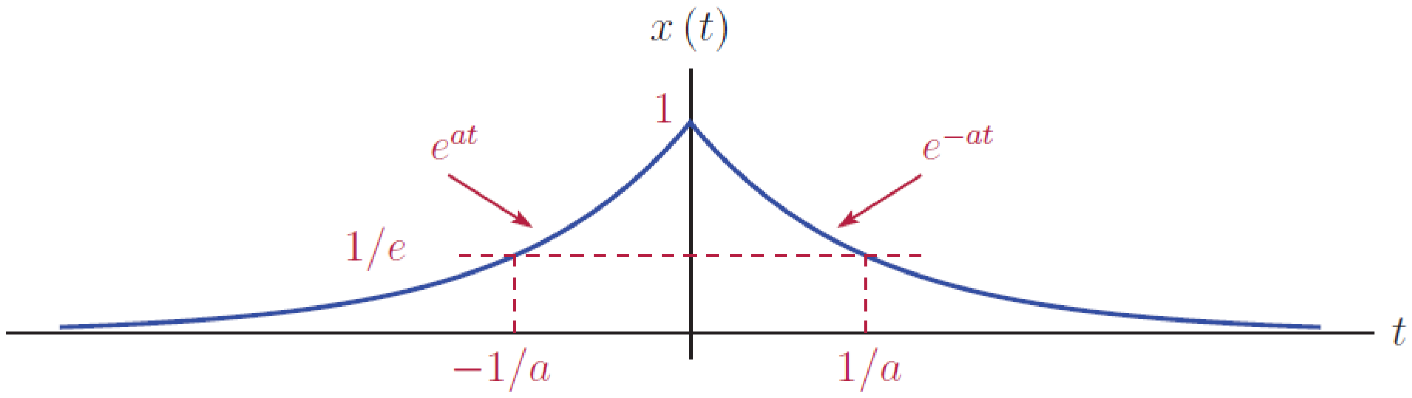


Figure 4.46 – Two-sided exponential signal $x(t)$ for Example 4.16.

Example 4.16 – Solution

$$X(\omega) = \int_{-\infty}^{\infty} e^{-a|t|} e^{-j\omega t} dt =$$

$$= \int_{-\infty}^{0} e^{at} e^{-j\omega t} dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t} dt = \int_{-\infty}^{0} e^{(a-j\omega)t} dt + \int_{0}^{\infty} e^{-(a+j\omega)t} dt$$

$$= \frac{1}{a-j\omega} e^{(a-j\omega)t} \Big|_{-\infty}^{0} + \frac{-1}{a+j\omega} e^{-(a+j\omega)t} \Big|_{0}^{\infty}$$

$$= \frac{1}{a-j\omega}[1-0] + \frac{-1}{a+j\omega}[0-1]$$

$$= \frac{1}{a-j\omega} + \frac{1}{a+j\omega} = \frac{a+j\omega+a-j\omega}{(a-j\omega)(a+j\omega)} = \frac{2a}{a^2-(j\omega)^2}$$

$$= \frac{2a}{a^2+\omega^2}$$

Example 4.16 – Book Solution

Solution: Applying the Fourier transform integral of Eqn. (4.127) to our signal we get

$$X(\omega) = \int_{-\infty}^{\infty} e^{-a|t|}\, e^{-j\omega t}\, dt$$

Recognizing that

$$t \le 0 \qquad \Rightarrow \quad e^{-a|t|} = e^{at}$$

$$t \ge 0 \qquad \Rightarrow \quad e^{-a|t|} = e^{-at}$$

the transform is

$$X(\omega) = \int_{-\infty}^{0} e^{at}\, e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t}\, dt$$

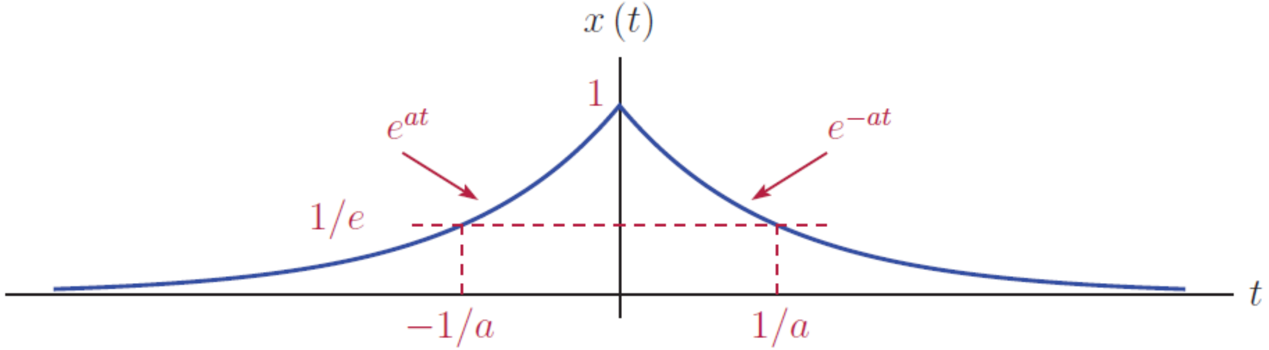
Example 4.16 – Book Solution

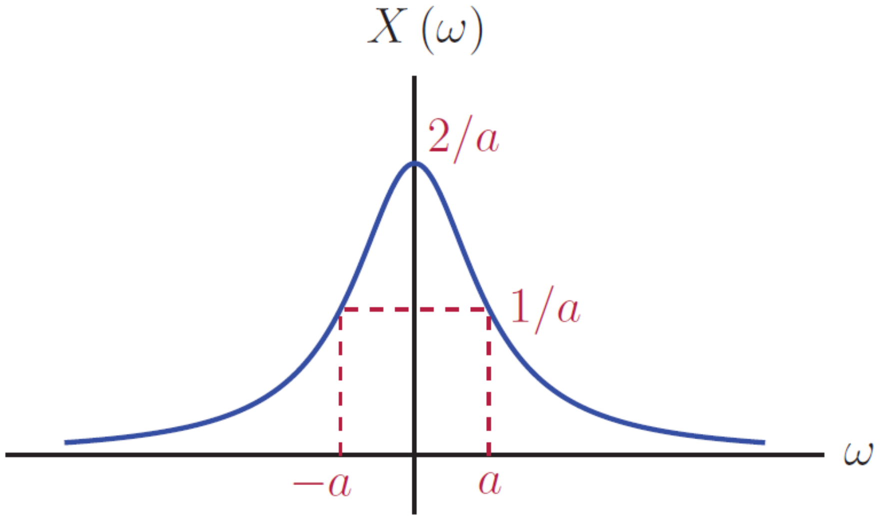
the transform is

$$X(\omega) = \int_{-\infty}^{0} e^{at}\, e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t}\, dt$$

$$= \frac{1}{a - j\omega} + \frac{1}{a + j\omega}$$

$$= \frac{2a}{a^2 + \omega^2}$$





Properties of the Fourier Transform: Linearity

- Fourier transform is a linear operator.

$$x_1(t) \overset{\mathcal{F}}{\longleftrightarrow} X_1(\omega)$$

$$x_2(t) \overset{\mathcal{F}}{\longleftrightarrow} X_2(\omega)$$

$$\alpha_1 x_1(t) + \alpha_2 x_2(t) \overset{\mathcal{F}}{\longleftrightarrow} \alpha_1 X_1(\omega) + \alpha_2 X_2(\omega)$$

Properties of the Fourier Transform: Linearity Proof

Proof: Using the forward transform equation given by Eqn. (4.127) with the time domain signal $[\alpha_1\, x_1(t) + \alpha_2\, x_2(t)]$ leads to:

$$\begin{aligned}\mathcal{F}\left\{\alpha_1\, x_1(t) + \alpha_2\, x_2(t)\right\} =& \int_{-\infty}^{\infty} \left[\alpha_1\, x_1(t) + \alpha_2\, x_2(t)\right] e^{-j\omega t}\, dt \\ =& \int_{-\infty}^{\infty} \alpha_1\, x_1(t)\; e^{-j\omega t}\, dt + \int_{-\infty}^{\infty} \alpha_2\, x_2(t)\; e^{-j\omega t}\, dt \\ =& \alpha_1 \int_{-\infty}^{\infty} x_1(t)\; e^{-j\omega t}\, dt + \alpha_2 \int_{-\infty}^{\infty} x_2(t)\; e^{-j\omega t}\, dt \\ =& \alpha_1\, \mathcal{F}\left\{x_1(t)\right\} + \alpha_2\, \mathcal{F}\left\{x_2(t)\right\}\end{aligned}$$

Properties of the Fourier Transform: Duality

- The transform relationship between $x(t)$ and $X(\omega)$ is defined by the inverse and forward Fourier transform integrals.

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) \quad \text{implies that} \quad X(t) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\, x(-\omega)$$

Properties of the Fourier Transform: Duality (using f instead of ω)

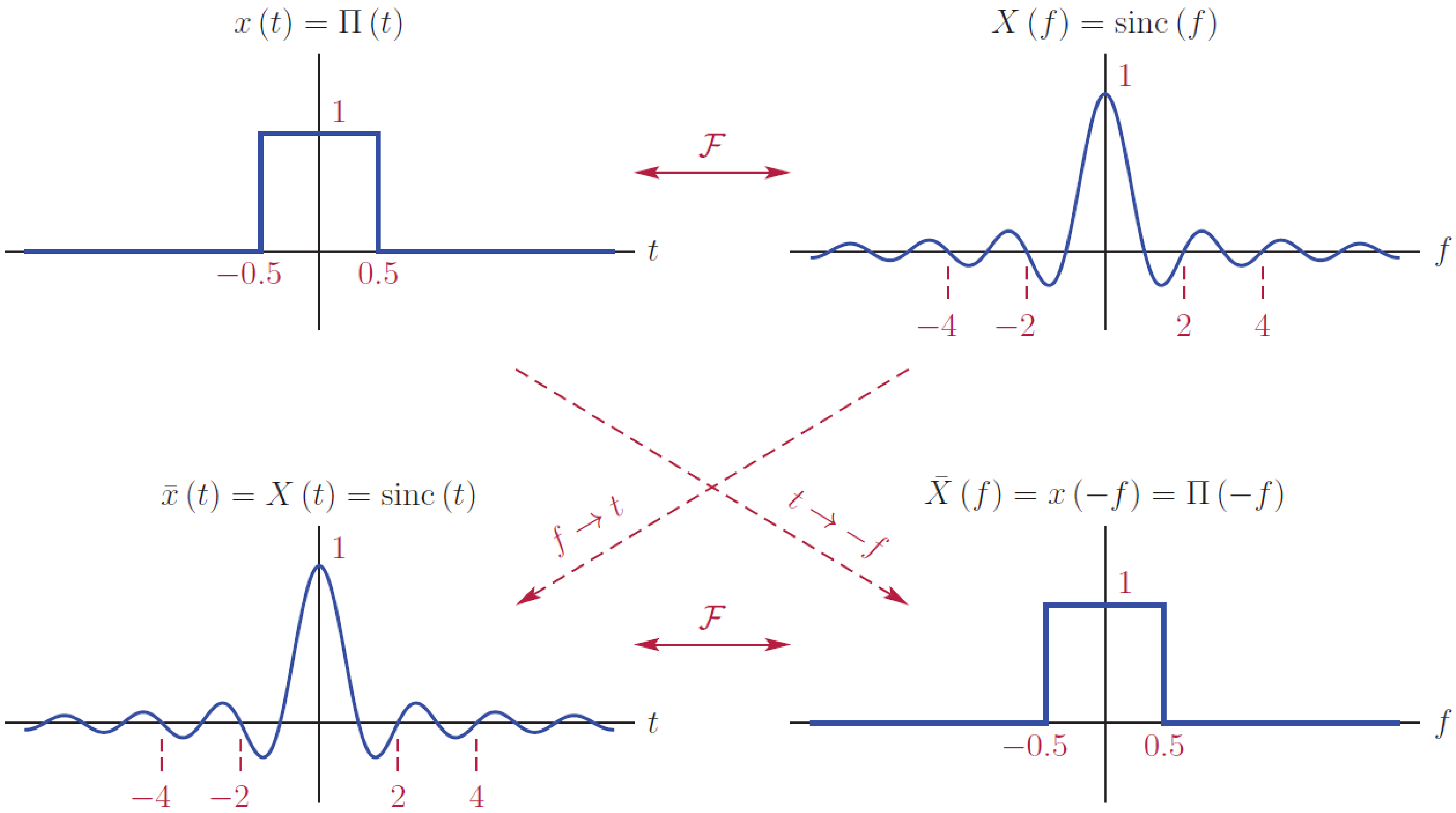
- The transform relationship between $x(t)$ and $X(\omega)$ is defined by the inverse and forward Fourier transform integrals.

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t}\, df$$

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(f) \quad \text{implies that} \quad X(t) \overset{\mathcal{F}}{\longleftrightarrow} x(-f)$$

Properties of the Fourier Transform: Duality (using f instead of ω)



Problem 4.24

4.24. The transform pair

$$e^{-a|t|} \overset{\mathcal{F}}{\longleftrightarrow} \frac{2a}{a^2+\omega^2}$$

Using this pair along with the duality property, find the Fourier transform of the signal

$$x(t) = \frac{2}{1+4t^2}$$

Problem 4.24 – Solution

Using the duality property we have

$$X(t) \overset{\mathscr{F}}{\longleftrightarrow} 2\pi\, x(-\omega)$$

or equivalently

$$\frac{2a}{a^2+t^2} \overset{\mathscr{F}}{\longleftrightarrow} 2\pi\, e^{-a|-\omega|}$$

Multiplying both the numerator and the denominator of the time-domain signal by 4 yields

$$\frac{8a}{4\,a^2+4\,t^2} \overset{\mathscr{F}}{\longleftrightarrow} 2\pi\, e^{-a|\omega|}$$

Let us choose

$$4\,a^2 = 1 \qquad \Rightarrow \qquad a = \frac{1}{2}$$

so that

$$\frac{4}{1+4\,t^2} \overset{\mathscr{F}}{\longleftrightarrow} 2\pi\, e^{-|\omega|/2}$$

Scaling both sides of the transform relationship by 1/2 we obtain the desired result:

$$\frac{2}{1+4\,t^2} \overset{\mathscr{F}}{\longleftrightarrow} \pi\, e^{-|\omega|/2}$$

Properties of the Fourier Transform: Time Shifting

For a transform pair

$$x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(\omega)$$

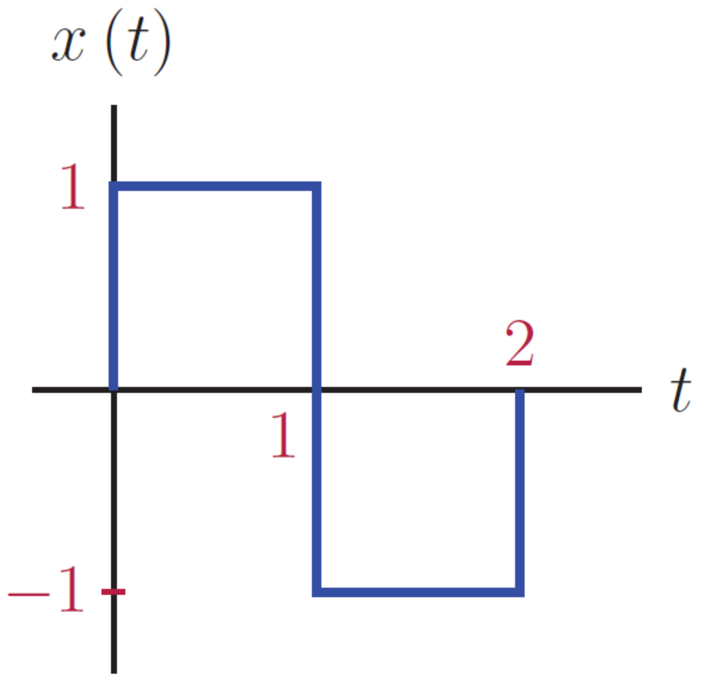
it can be shown that

$$x(t-\tau) \stackrel{\mathcal{F}}{\longleftrightarrow} X(\omega)\, e^{-j\omega\tau}$$

Problem 4.21

4.21. Refer to the signal shown in Fig. P.4.19. Find its Fourier transform by starting with the transform of the unit pulse and using linearity and time shifting properties.

$$\Pi(t-0.5)-\Pi(t-1.5)$$



Problem 4.21 – Solution

Using the unit-pulse function $\Pi(t)$ we have

$$\mathscr{F}\{\Pi(t-0.5)\} = \text{sinc}(f)\, e^{-j\pi f}$$

and

$$\mathscr{F}\{\Pi(t-1.5)\} = \text{sinc}(f)\, e^{-j3\pi f}$$

Utilizing linearity of the Fourier transform

$$\mathscr{F}\{\Pi(t-0.5) - \Pi(t-1.5)\} = \text{sinc}(f)\left[e^{-j\pi f} - e^{-j3\pi f}\right]$$

Properties of the Fourier Transform: Frequency Shifting

For a transform pair

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$$

it can be shown that

$$x(t)\, e^{j\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} X(\omega - \omega_0)$$

Properties of the Fourier Transform: Modulation Property

For a transform pair

$$x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(\omega)$$

it can be shown that

$$x(t)\cos(\omega_0 t) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{2}\left[X(\omega-\omega_0)+X(\omega+\omega_0)\right]$$

and

$$x(t)\sin(\omega_0 t) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{2}\left[X(\omega-\omega_0)\,e^{-j\pi/2}+X(\omega+\omega_0)\,e^{j\pi/2}\right]$$

Properties of the Fourier Transform: Time and Frequency Scaling

For a transform pair

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$$

it can be shown that

$$x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

The parameter a is any non-zero and real-valued constant.

Properties of the Fourier Transform: Convolution Property

For two transform pairs

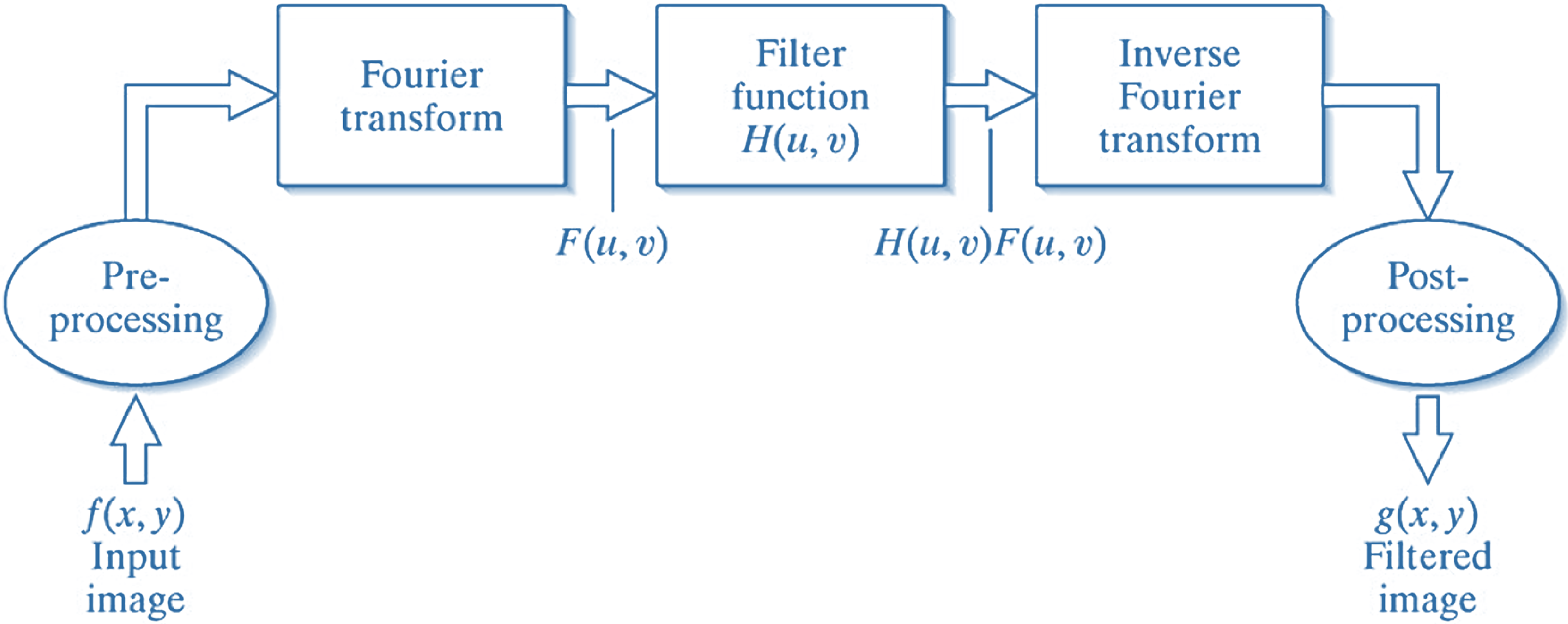
$$x_1(t) \overset{\mathcal{F}}{\longleftrightarrow} X_1(\omega) \quad \text{and} \quad x_2(t) \overset{\mathcal{F}}{\longleftrightarrow} X_2(\omega)$$

it can be shown that

$$x_1(t) * x_2(t) \overset{\mathcal{F}}{\longleftrightarrow} X_1(\omega)\, X_2(\omega)$$

Properties of the Fourier Transform: Convolution Property

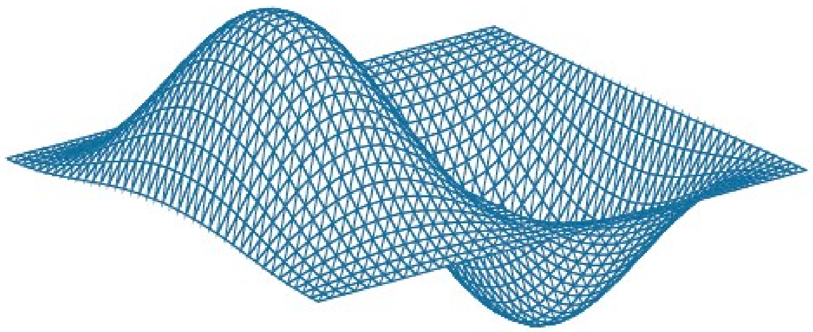
$$(f \star h)(x,y) \Leftrightarrow (F \bullet H)(u,v)$$

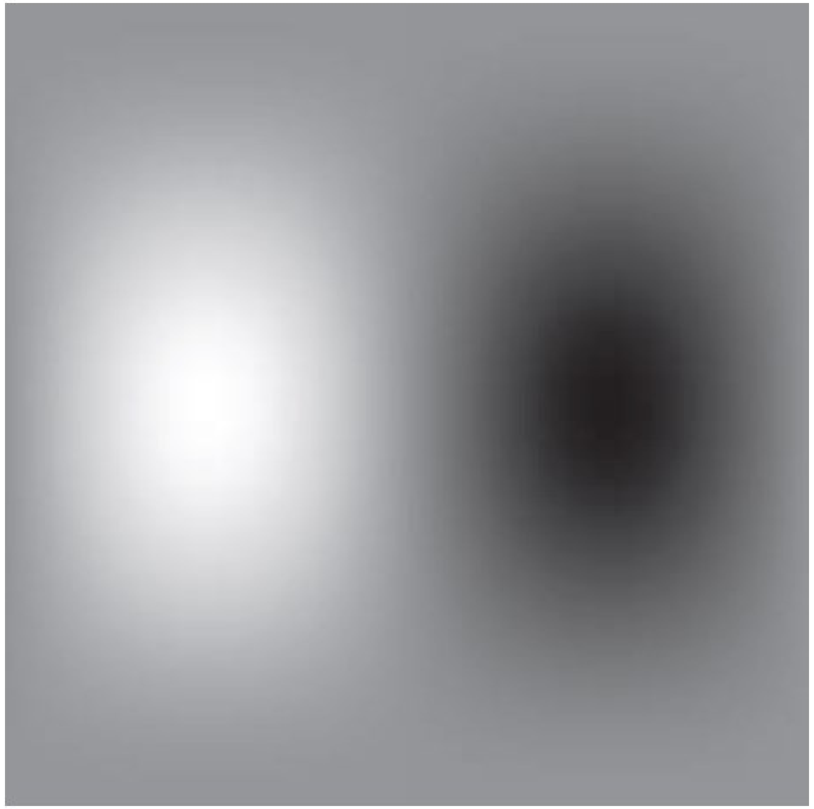


Properties of the Fourier Transform: Convolution Property

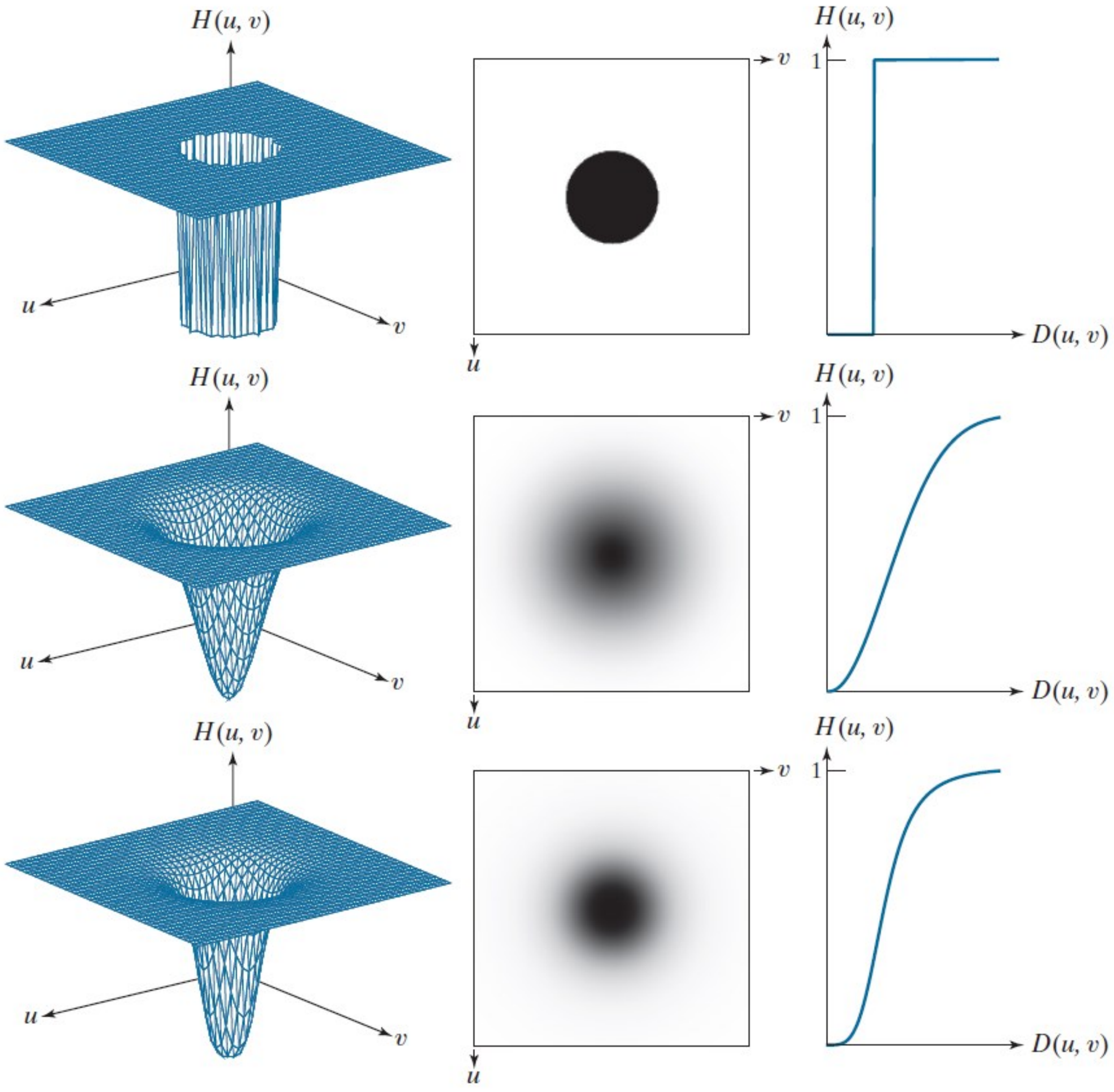
$$(f \star h)(x,y) \Leftrightarrow (F \bullet H)(u,v)$$

−1	0	1
−2	0	2
−1	0	1





Properties of the Fourier Transform: Convolution Property



Properties of the Fourier Transform: Multiplication of Two Signals

For two transform pairs

$$x_1(t) \overset{\mathcal{F}}{\longleftrightarrow} X_1(\omega) \quad \text{and} \quad x_2(t) \overset{\mathcal{F}}{\longleftrightarrow} X_2(\omega)$$

it can be shown that

$$x_1(t)\, x_2(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi} X_1(\omega) * X_2(\omega)$$

If we choose to use f instead of ω, then

$$x_1(t)\, x_2(t) \overset{\mathcal{F}}{\longleftrightarrow} X_1(f) * X_2(f)$$

Fourier Transforms of Some Basic Signals

Name	Signal	Transform
Rectangular pulse	$x(t) = A\,\Pi(t/\tau)$	$X(\omega) = A\tau \text{ sinc}\left(\frac{\omega\tau}{2\pi}\right)$
Triangular pulse	$x(t) = A\,\Lambda(t/\tau)$	$X(\omega) = A\tau \text{ sinc}^2\left(\frac{\omega\tau}{2\pi}\right)$
Right-sided exponential	$x(t) = e^{-at}\,u(t)$	$X(\omega) = \frac{1}{a + j\omega}$
Two-sided exponential	$x(t) = e^{-a\lvert t\rvert}$	$X(\omega) = \frac{2a}{a^2 + \omega^2}$
Signum function	$x(t) = \text{sgn}(t)$	$X(\omega) = \frac{2}{j\omega}$
Unit impulse	$x(t) = \delta(t)$	$X(\omega) = 1$
Sinc function	$x(t) = \text{sinc}(t)$	$X(\omega) = \Pi\left(\frac{\omega}{2\pi}\right)$
Constant-amplitude signal	$x(t) = 1,\ \text{all } t$	$X(\omega) = 2\pi\,\delta(\omega)$
	$x(t) = \frac{1}{\pi t}$	$X(\omega) = -j \text{ sgn}(\omega)$
Unit-step function	$x(t) = u(t)$	$X(\omega) = \pi\,\delta(\omega) + \frac{1}{j\omega}$
Modulated pulse	$x(t) = \Pi\left(\frac{t}{\tau}\right)\cos(\omega_0 t)$	$X(\omega) = \frac{\tau}{2} \text{ sinc}\left(\frac{(\omega - \omega_0)\tau}{2\pi}\right) + \frac{\tau}{2} \text{ sinc}\left(\frac{(\omega + \omega_0)\tau}{2\pi}\right)$

Fourier Transform Properties

Property	Signal	Transform
Linearity	$\alpha\, x_1(t) + \beta\, x_2(t)$	$\alpha\, X_1(\omega) + \beta\, X_2(\omega)$
Duality	$X(t)$	$2\pi\, x(-\omega)$
Conjugate symmetry	$x(t)$ real	$X^*(\omega) = X(-\omega)$ Magnitude: $\lvert X(-\omega)\rvert = \lvert X(\omega)\rvert$ Phase: $\Theta(-\omega) = -\Theta(\omega)$ Real part: $X_r(-\omega) = X_r(\omega)$ Imaginary part: $X_i(-\omega) = -X_i(\omega)$
Conjugate antisymmetry	$x(t)$ imaginary	$X^*(\omega) = -X(-\omega)$ Magnitude: $\lvert X(-\omega)\rvert = \lvert X(\omega)\rvert$ Phase: $\Theta(-\omega) = -\Theta(\omega) \mp \pi$ Real part: $X_r(-\omega) = -X_r(\omega)$ Imaginary part: $X_i(-\omega) = X_i(\omega)$
Even signal	$x(-t) = x(t)$	$\mathrm{Im}\{X(\omega)\} = 0$
Odd signal	$x(-t) = -x(t)$	$\mathrm{Re}\{X(\omega)\} = 0$
Time shifting	$x(t-\tau)$	$X(\omega)\, e^{-j\omega\tau}$
Frequency shifting	$x(t)\, e^{j\omega_0 t}$	$X(\omega - \omega_0)$
Modulation property	$x(t)\, \cos(\omega_0 t)$	$\frac{1}{2}\left[X(\omega-\omega_0) + X(\omega+\omega_0)\right]$
Time and frequency scaling	$x(at)$	$\dfrac{1}{\lvert a\rvert} X\left(\dfrac{\omega}{a}\right)$
Differentiation in time	$\dfrac{d^n}{dt^n}\left[x(t)\right]$	$(j\omega)^n\, X(\omega)$
Differentiation in frequency	$(-jt)^n\, x(t)$	$\dfrac{d^n}{d\omega^n}\left[X(\omega)\right]$
Convolution	$x_1(t) * x_2(t)$	$X_1(\omega)\, X_2(\omega)$
Multiplication	$x_1(t)\, x_2(t)$	$\dfrac{1}{2\pi} X_1(\omega) * X_2(\omega)$
Integration	$\int_{-\infty}^{t} x(\lambda)\, d\lambda$	$\dfrac{X(\omega)}{j\omega} + \pi\, X(0)\, \delta(\omega)$
Parseval's theorem	$\int_{-\infty}^{\infty} \lvert x(t)\rvert^2\, dt = \dfrac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\omega)\rvert^2\, d\omega$	

Energy and Power in the Frequency Domain

- We will discuss a very important theorem of Fourier series and transform known as Parseval's theorem.
- Parseval's theorem can be used as the basis of computing energy or power of a signal from its frequency domain representation.

Parseval's Theorem

For a periodic power signal $\tilde{x}(t)$ with period of T_0 and EFS coefficients $\{c_k\}$ it can be shown that

$$\frac{1}{T_0}\int_{t_0}^{t_0+T_0} \left|\tilde{x}(t)\right|^2 \, dt = \sum_{k=-\infty}^{\infty} \left|c_k\right|^2$$

For a non-periodic energy signal $x(t)$ with a Fourier transform $X(f)$, the following holds true:

$$\int_{-\infty}^{\infty} \left|x(t)\right|^2 \, dt = \int_{-\infty}^{\infty} \left|X(f)\right|^2 \, df$$

Energy and Power Spectral Density

- Power spectral density of a periodic signal

$$S_x(f) = \sum_{k=-\infty}^{\infty} |c_k|^2 \, \delta(f - kf_0)$$

- Energy spectral density of a non-periodic signal

$$G_x(f) = |X(f)|^2$$

Problem 4.38

4.38. Determine and sketch the power spectral density of the following signals:

a. $x(t) = 3\cos(20\pi t)$

b. $x(t) = 2\cos(20\pi t) + 3\cos(30\pi t)$

c. $x(t) = 5\cos(200\pi t) + 5\cos(200\pi t)\cos(30\pi t)$

Problem 4.38 (a) – Solution

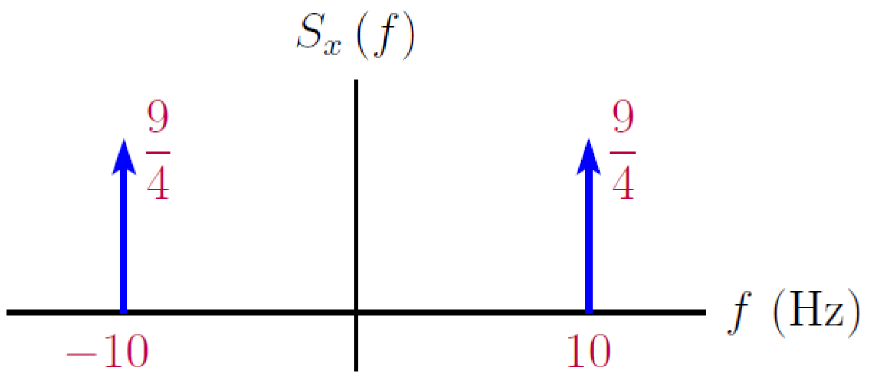
a. $x(t) = 3\cos(20\pi t)$

a. For the signal $x(t)$ the fundamental frequency is $f_0 = 10$ Hz, and the EFS coefficients are

$$c_k = \begin{cases} \dfrac{3}{2}, & k = \pm 1 \\ 0, & \text{otherwise} \end{cases}$$

The power spectral density is

$$S_x(f) = \frac{9}{4}\delta(f+10) + \frac{9}{4}\delta(f-10)$$



Problem 4.38 (b) – Solution

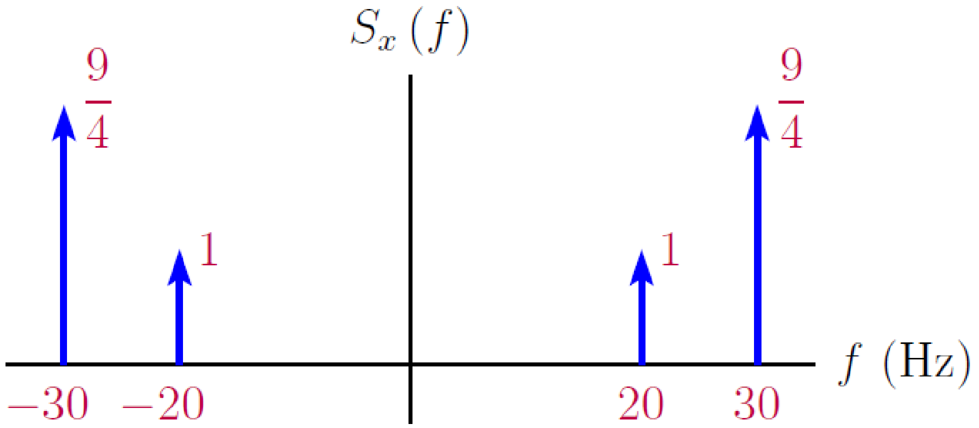
b. $x(t) = 2\cos(20\pi t) + 3\cos(30\pi t)$

b. For the signal $x(t)$ the fundamental frequency is $f_0 = 10$ Hz, and the EFS coefficients are

$$c_k = \begin{cases} 1\,, & k = \pm 2 \\ \dfrac{3}{2}\,, & k = \pm 3 \\ 0\,, & \text{otherwise} \end{cases}$$

The power spectral density is

$$S_x(f) = \frac{9}{4}\delta(f+30) + \delta(f+20) + \delta(f-20) + \frac{9}{4}\delta(f-30)$$

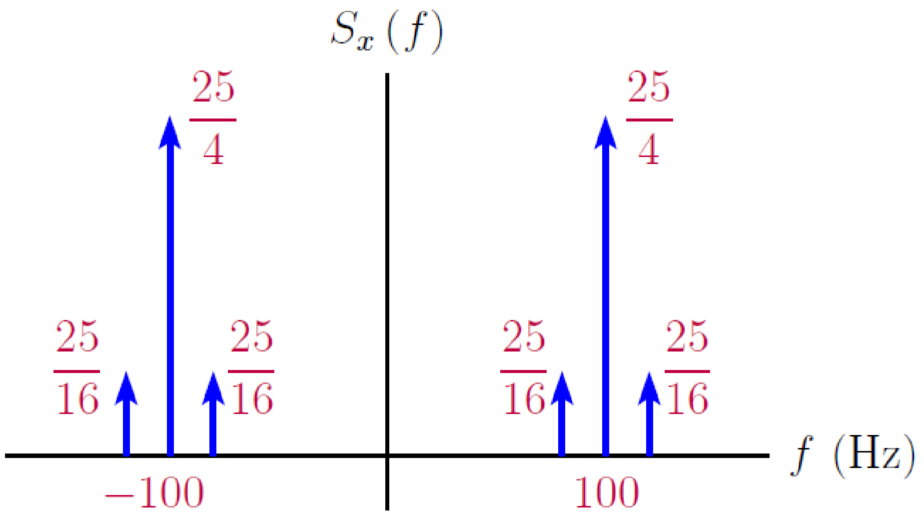


Problem 4.38 (c) – Solution

c. $$x(t) = 5\cos(200\pi t) + 5\cos(200\pi t)\cos(30\pi t)$$

c. For the signal $x(t)$ the fundamental frequency is $f_0 = 5$ Hz, and the EFS coefficients are

$$c_k = \begin{cases} \frac{5}{4}, & k = \pm 17 \\ \frac{5}{2}, & k = \pm 20 \\ \frac{5}{4}, & k = \pm 23 \\ 0, & \text{otherwise} \end{cases}$$



The power spectral density is

$$S_x(f) = \frac{25}{16}\delta(f+230) + \frac{25}{4}\delta(f+200) + \frac{25}{16}\delta(f+170) + \frac{25}{16}\delta(f-170) + \frac{25}{4}\delta(f-200) + \frac{25}{16}\delta(f-230)$$

Example 4.39

Example 4.39: Power spectral density of a sinusoidal signal

Find the power spectral density of the signal $\tilde{x}(t) = 5\cos(200\pi t)$.

Example 4.39 – Solution

Solution: Using Euler's formula, the signal in question can be written as

$$x(t) = \frac{5}{2}\, e^{-j200\pi t} + \frac{5}{2}\, e^{j200\pi t}$$

from an inspection of which we conclude that the only significant coefficients in the EFS representation of the signal are

$$c_{-1} = c_1 = \frac{5}{2}$$

with all other coefficients equal to zero. The fundamental frequency is $f_0 = 100$ Hz. Using Eqn. (4.315), the power spectral density is

$$\begin{aligned} S_x(f) &= \sum_{n=-\infty}^{\infty} |c_n|^2\, \delta(f - 100n) \\ &= |c_{-1}|^2\, \delta(f + 100) + |c_1|^2\, \delta(f - 100) \\ &= \frac{25}{4}\, \delta(f + 100) + \frac{25}{4}\, \delta(f - 100) \end{aligned}$$

Example 4.39 – Solution

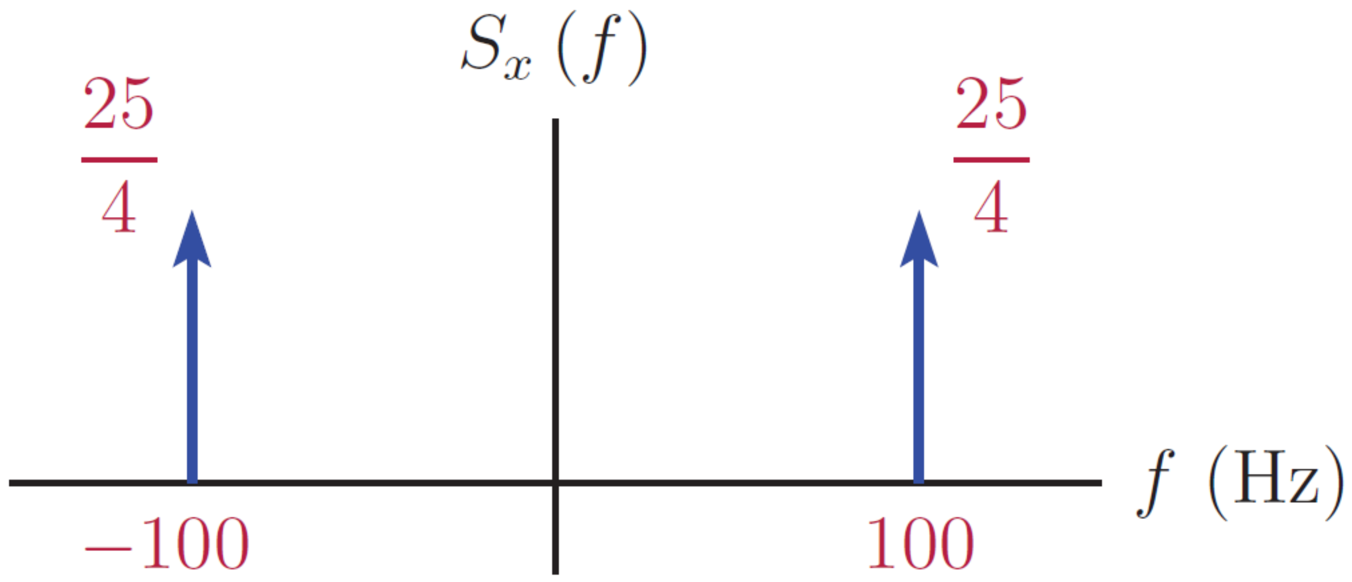
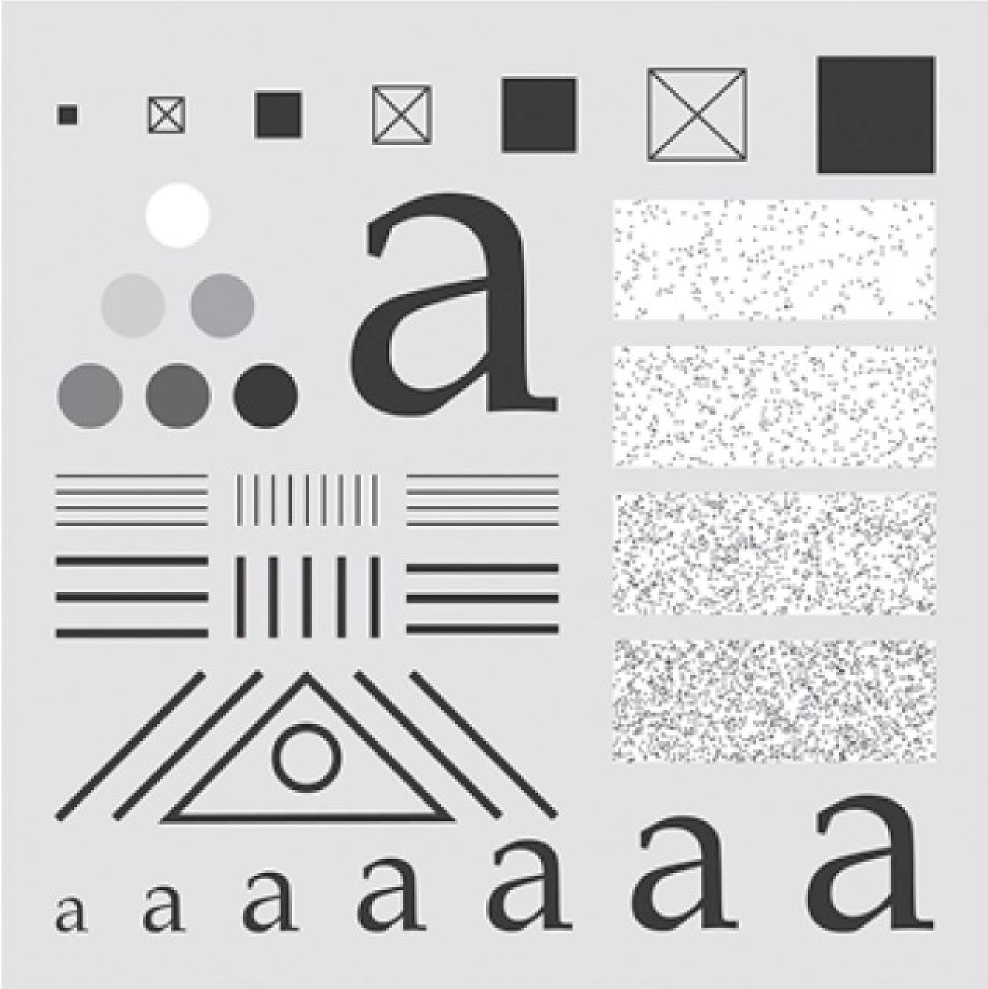
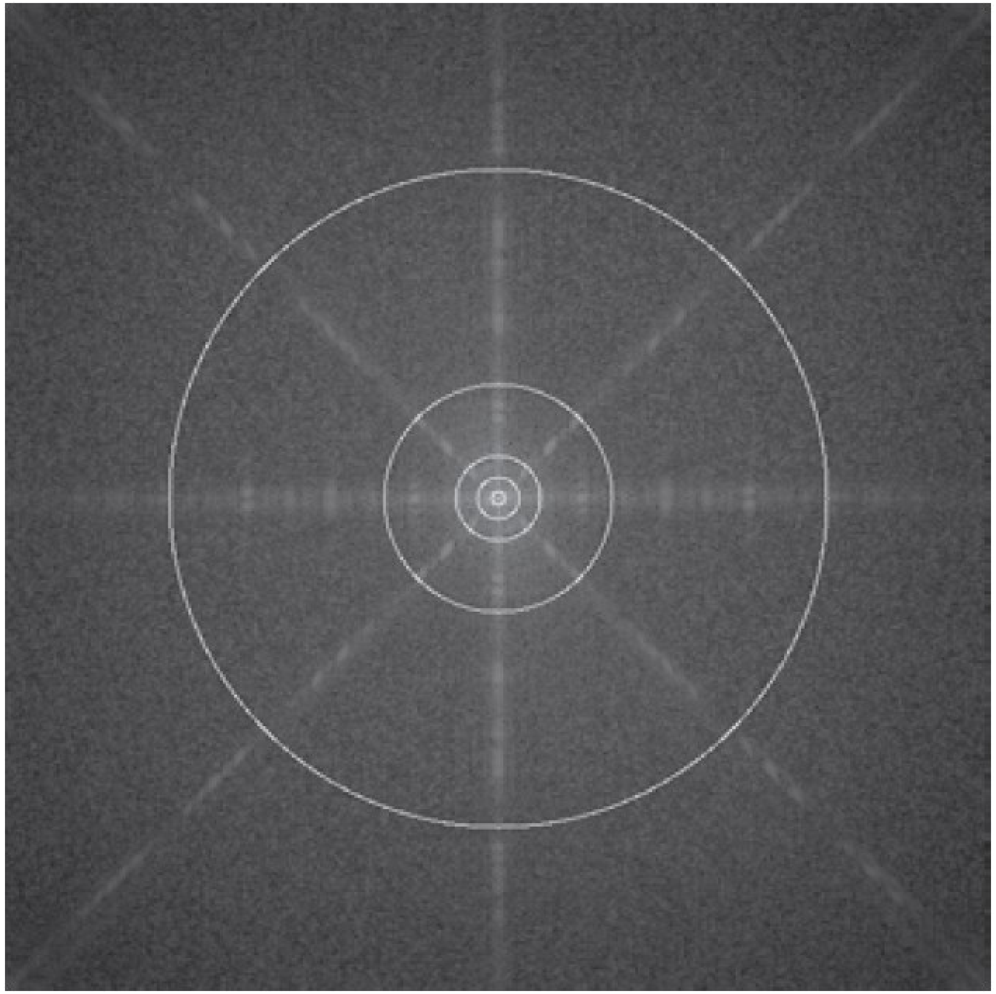


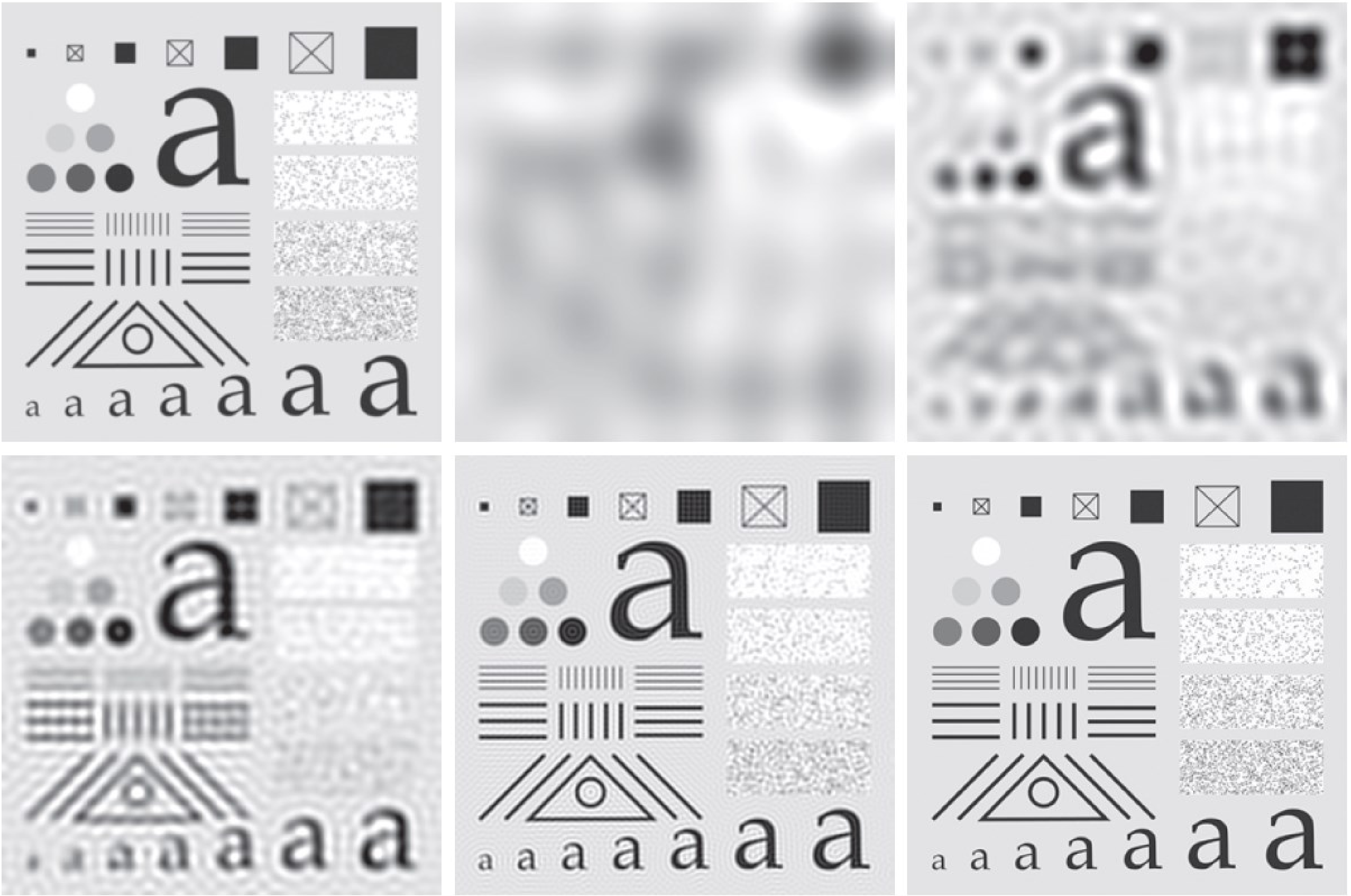
Figure 4.83 – Power spectral density

Filtering in the Frequency Domain

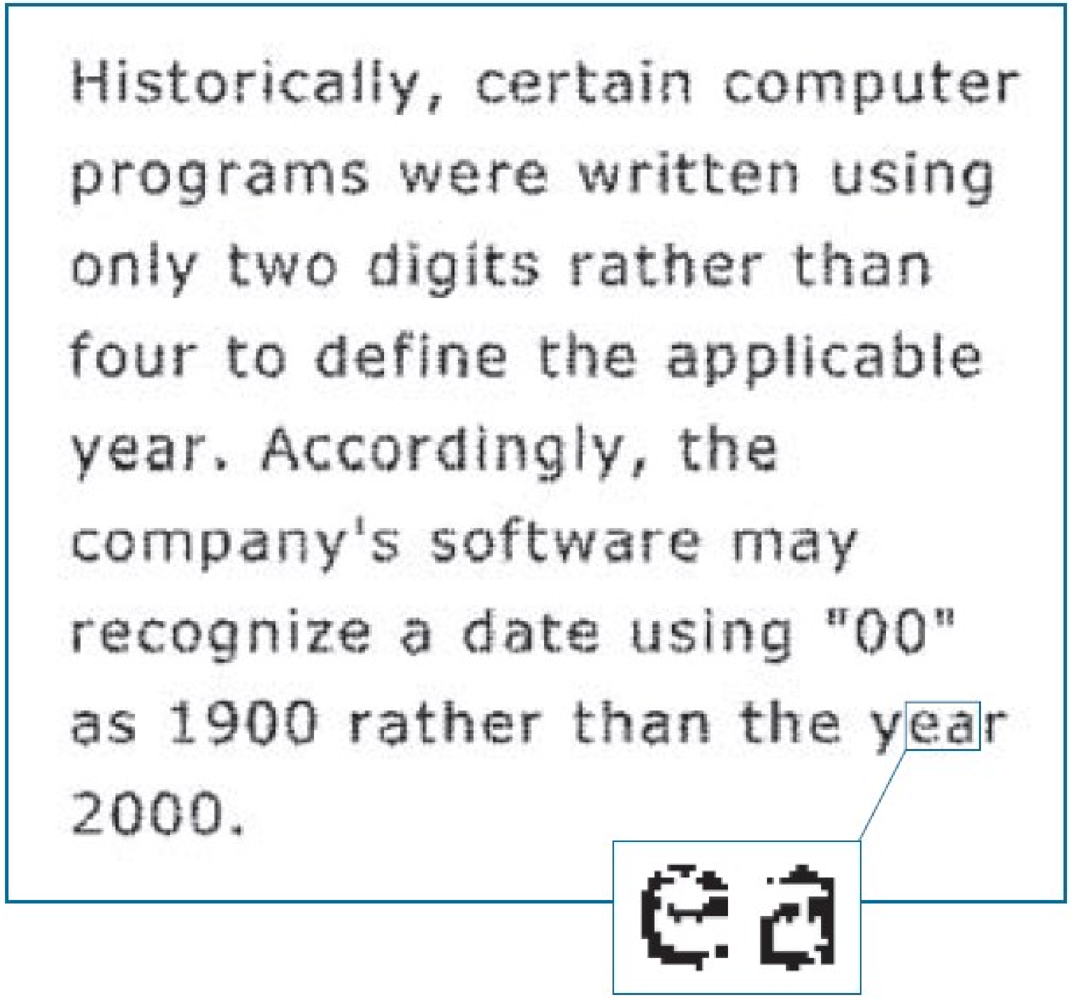


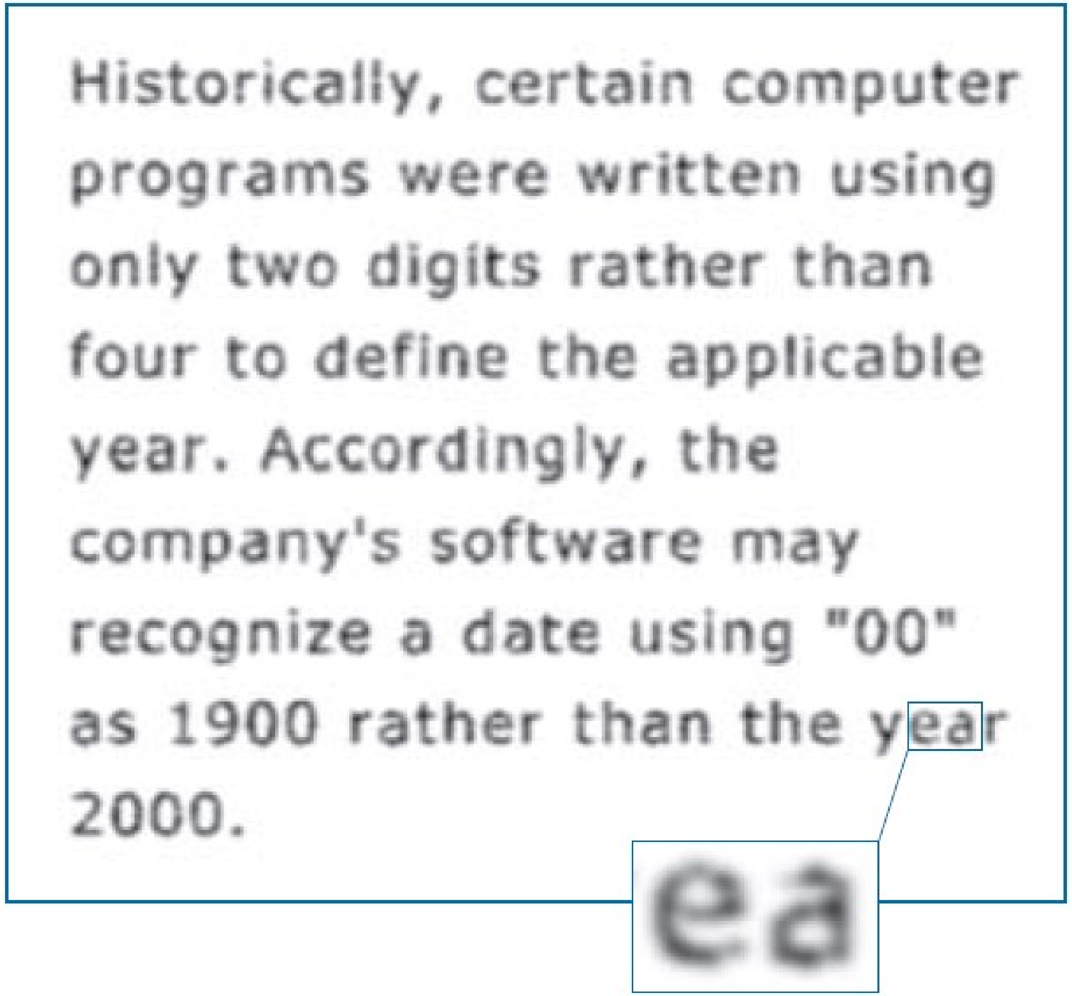


Filtering in the Frequency Domain: Lowpass Filters



Filtering in the Frequency Domain: Lowpass Filters





Filtering in the Frequency Domain: Highpass Filters

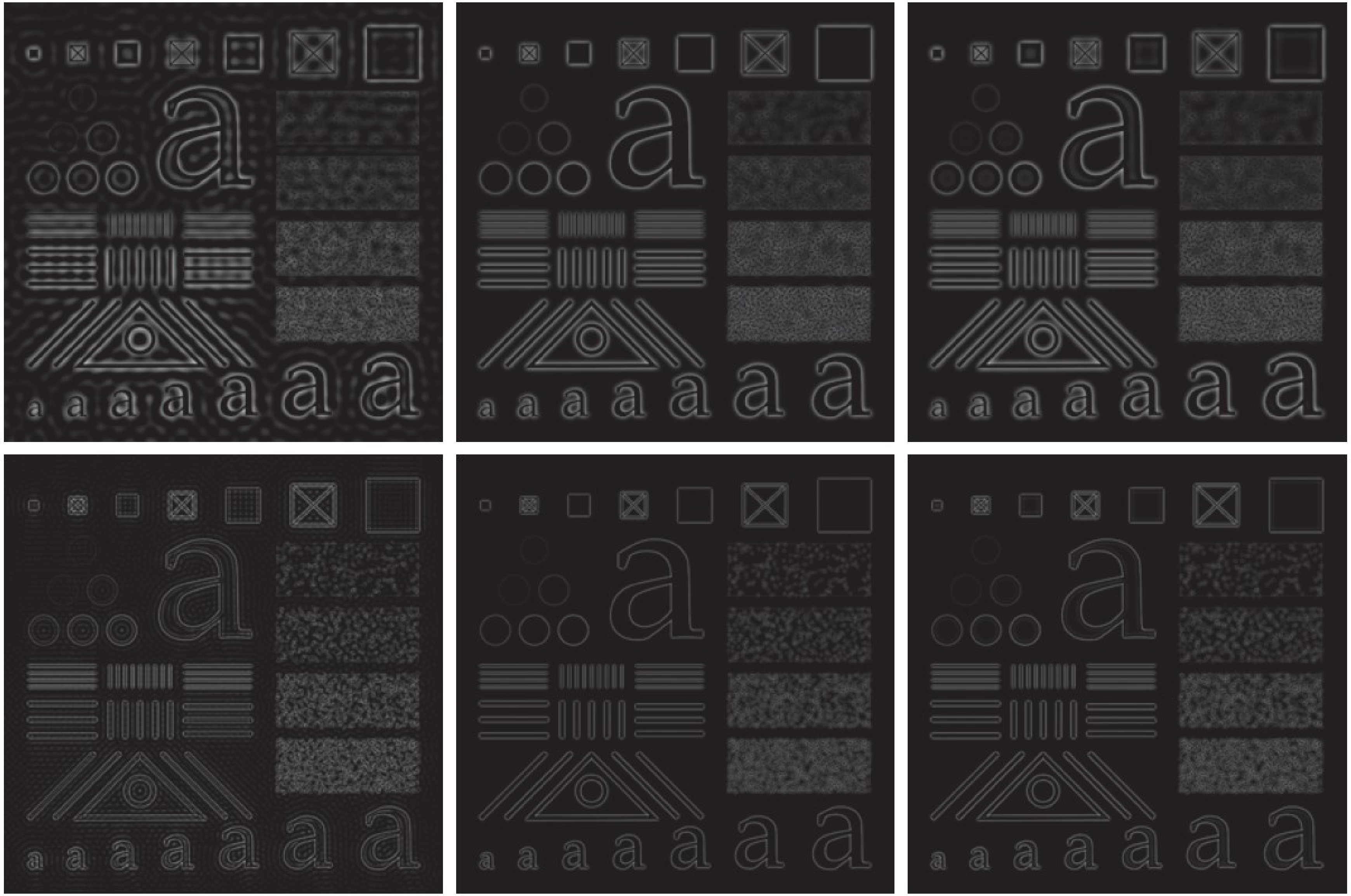
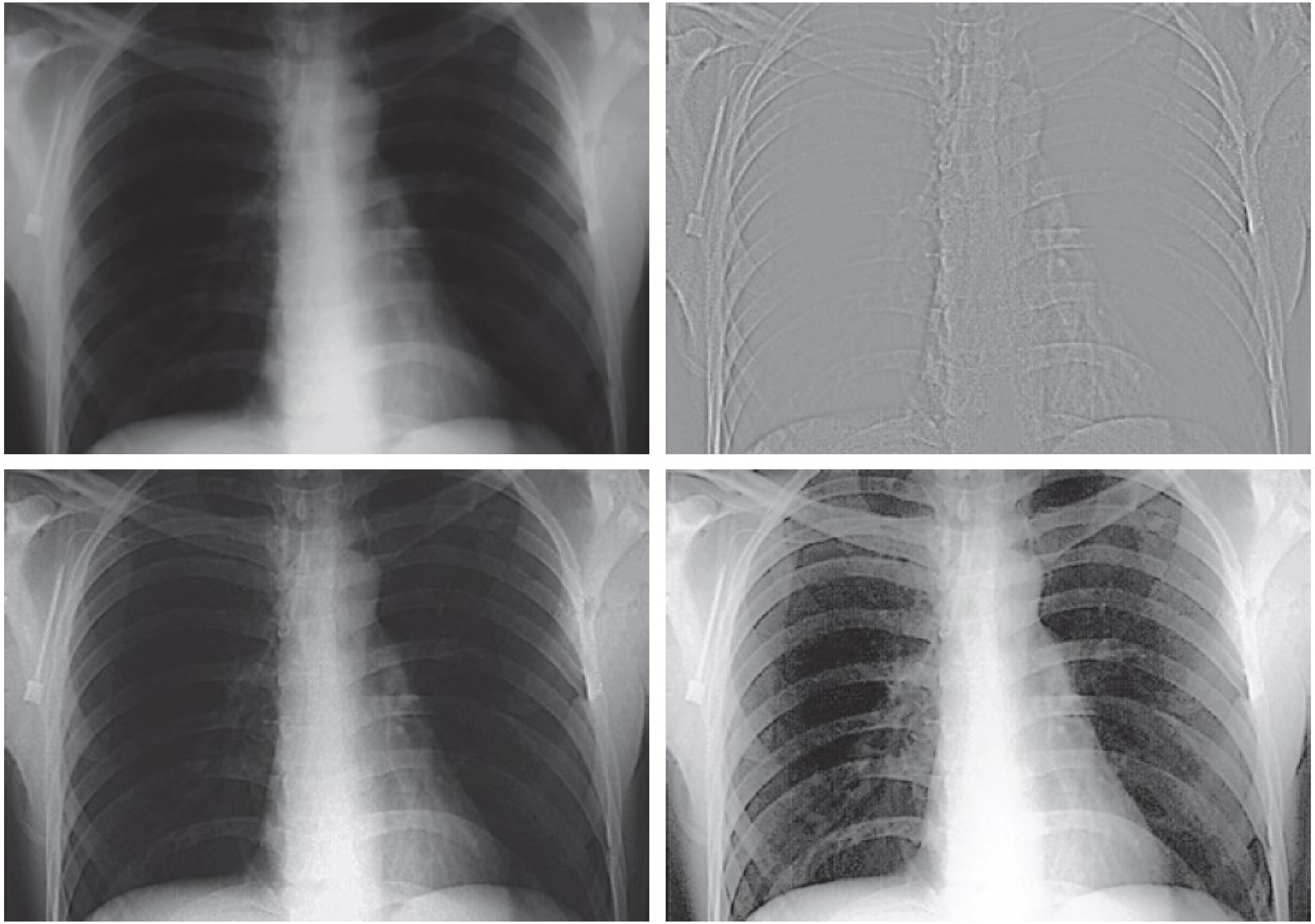


Image Enhancement Using the Laplacian in the Frequency Domain

Image Enhancement in the Frequency Domain



Periodic Noise Reduction Using Frequency Domain